Math 219, Homework 2

Due date: 23.11.2005, Wednesday

This homework concerns two (fictitious) design problems about the solar car "MEŞ-e" of the METU Robotics Society, which won the Formula-G trophy in September 2005. Just for the purposes of this homework, assume that they want to modify the car, and they are asking for your help on two issues.

1. The first problem is about the shock absorbing system of the car. We may model the shock absorber as a single linear spring. This question concerns how to adjust the damping coefficient in order to meet certain requirements.

(a) It is known that when the pilot, weighing $80kg$, gets into the car seat, the shock absorber is compressed by $5cm$. From this data, compute the spring constant k (in kg/sec^2).

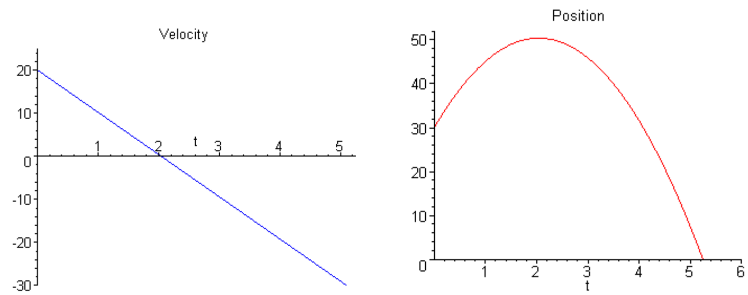
(b) The car (without the pilot) weighs $240kg$. Write a differential equation which governs the vertical motion of the car (this could for instance describe the vertical

$20(a)$. The concentration is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $c(t{\to}\infty) = k + P/r$.

(b). $c(t) = c_0\, e^{-rt/V}$. The *reduction times* are $T_{50} = ln(2)V/r$ and $T_{10} = ln(10)V/r$.

(c). The *reduction times*, in *years*, are $T_S = ln(10)(65.2)/12,200 = 430.85$
$T_M = ln(10)(158)/4,900 = 71.4$; $T_E = ln(10)(175)/460 = 6.05$
$T_O = ln(10)(209)/16,000 = 17.63$.

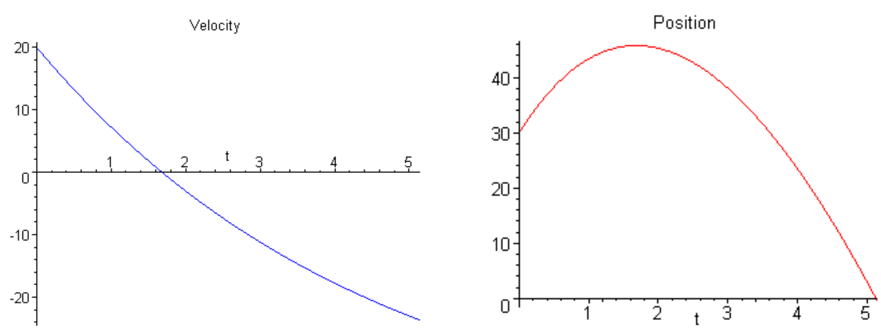
$21(c)$.



$22(a)$. The differential equation for the motion is $m\, dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ *m/s*, the solution is $v(t) = -44.1 + 64.1\, exp(-t/4.5)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ *sec*. Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\, t - 288.45\, exp(-t/4.5)$. Hence the *maximum height* is $x(t_1) = 45.78$ *m*.

(b). Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ *sec*.

(c).



$23(a)$. The differential equation for the *upward* motion is $m\, dv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is *separable*, with $\frac{m}{\mu v^2 + mg} dv = -dt$. Integrating

(b). Substituting the results in Part(a) into the general ODE, $y'' + p(t)y' + q(t)y = 0$, `we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c). Assuming $\left[\frac{dx}{dt}\right]^2 = k\,q(t)$, and $q(t) > 0$, we find that $\frac{dx}{dt} = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = \xi(t) = \int\sqrt{k\,q(t)}\;dt$.

(d). Let $k = 1$. It follows that $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = \frac{d\xi}{dt} + p(t)\xi(t) = \frac{q'}{2\sqrt{q}} + p\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\Big/\left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0\,.$$

* For the case $q(t) < 0$, write $q(t) = -[-q(t)]$, and set $\left[\frac{dx}{dt}\right]^2 = -q(t)$.

36. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \left(1 + 3t^2\right)/t^2\,.$$

The ratio is *not* constant, and therefore the equation cannot be transformed.

37. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Prob. 35, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution is

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \cdots$$

The *nearest* zero of $P(x) = cos\, x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $ln(1+x)$, about $x_0 = 0$, is

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + \\ + \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right] \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

The *first* product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \cdots.$$

The *second* product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \cdots.$$

Combining the series and equating the coefficients to *zero*, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1 \frac{7x^5}{120} + \left(\frac{5}{3}a_1 - a_0\right)\frac{x^6}{120} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \cdots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \cdots$$

The coefficient $p(x) = e^x ln(1+x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-\left(\frac{1}{4}+i\right)t} \\ &= \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4}(\cos t - i\,\sin t) \\ &= e^{-t/4}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + i e^{-t/4}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.\end{aligned}$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

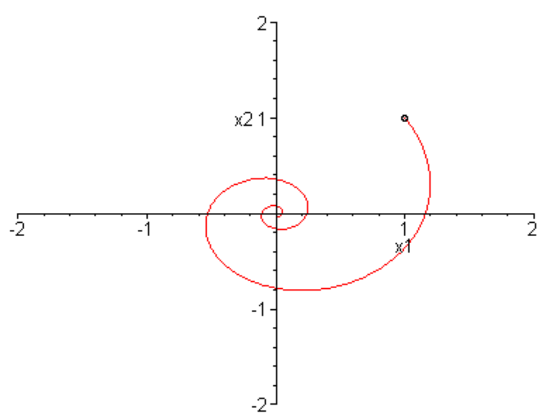
$$\mathbf{x} = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2\, e^{-t/4}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_3\, e^{-t/4}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

(b). Let $\mathbf{x}(0) = (x_1^0\,, x_2^0\,, x_3^0)$. The solution can be written as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0\, e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} x_2^0\,\sin t + x_1^0\,\cos t \\ x_2^0\,\cos t - x_1^0\,\sin t \\ 0 \end{pmatrix}.$$

With $\mathbf{x}(0) = (1\,, 1\,, 1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} \sin t + \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix}.$$



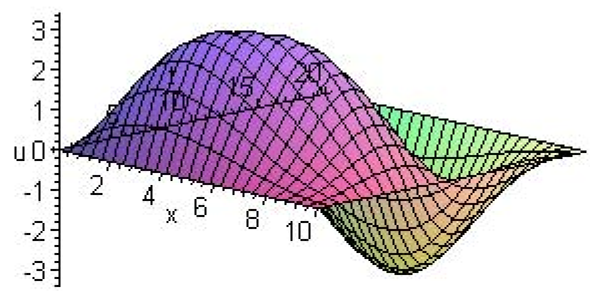
We use the starting values from the *exact solution* :

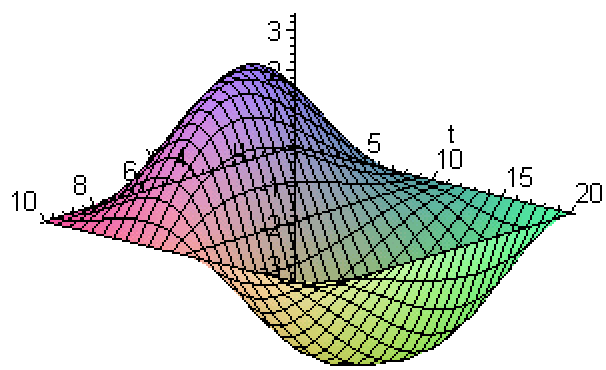
	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0	0.1	0.2	0.3
x_n	1.0	1.12883	1.32042	1.60021
y_n	0.0	-0.11057	-0.250847	-0.429696

One time step using the *predictor-corrector* method results in the approximate values:

	$n = 4(pre)$	$n = 4(cor)$
t_n	0.4	0.4
x_n	1.99445	1.99521
y_n	-0.662064	-0.662442

(d).





$8(a)$. As given by Eq. (34), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}k_n\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0)=g(x)$. It follows that