solutions manual for

classical electromagnetic radiation

3 edition

Heald

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is $K_b = I'/d$, which equals cM by Eq. (1.58). If the physical boundary is tilted so that **n** remains perpendicular to **M**, the cell can be deformed to match with no change in K_b. But when the boundary normal **n** tilts out of the plane perpendicular to **M**, and the cell wall deformed to match, the same current I' is distributed over a width that is larger by $1/\cos\theta$, where θ is the angle between **n** and the plane perpendicular to **M**. Thus the surface current-density is reduced by the *sine* of the angle between **n** and **M**, and $-\mathbf{n}\times\mathbf{M}$ properly represents the current density including its vector sense.

1-12. From Eqs. (1.25) and (A.50),

$$\rho_b \;=\; -\,\mathrm{div}\;\mathbf{P}(r) \;=\; -\frac{1}{r^2}\frac{d}{dr}[r^2(kr)] \;=\; 3k$$

At the surface $r = a$, by Eq. (1.34) or Prob. 1-11, there is a surface charge density

$$(\rho_s)_b \;=\; \mathbf{e}_r \cdot \mathbf{P}(a) \;=\; ka$$

We have spherical symmetry and can use Gauss' law. Using a spherical Gaussian surface of radius r with Eq. (1.6),

$$\oint \mathbf{E}\cdot d\mathbf{a} \;\rightarrow\; E_r(r)\,4\pi r^2 \;=\; 4\pi q_{\mathrm{encl}}$$

When $r < a$, the charge enclosed is

$$q = \int \rho_b\,dv \;=\; -\,3k\int_0^r 4\pi r^2\,dr \;=\; -\,4\pi k r^3$$

$$\mathbf{E}(r<a) \;=\; -\,4\pi k r\,\mathbf{e}_r \;=\; -\,4\pi\mathbf{P}$$

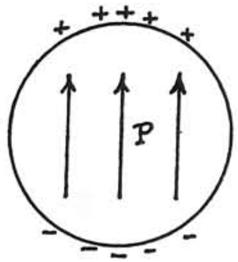
When $r > a$, the charge enclosed is zero. (The dielectric dipoles consist of equal positive and negative charge. Formally, the volume integral of ρ is canceled by the surface integral of ρ_s.) Thus, $\mathbf{E}(r>a) = 0$.

Gauss' law for the **D** field, Eq. (1.82), gives

$$\oint \mathbf{D}\cdot d\mathbf{a} \;=\; 4\pi q_{\mathrm{free}} \;\rightarrow\; 0$$

That is, $\mathbf{D} = 0$ inside as well as out, since no free charge is present. As a check, note that Eq. (1.28) is satisfied both inside and outside the sphere.

1-13. **(a)** The bound surface charge is



$$(\rho_s)_b \;=\; \mathbf{n}\cdot\mathbf{P}_0 \;=\; P_0\cos\theta$$

where θ is the polar angle with respect to the z axis. By symmetry, the net field at the center (the origin) will be in the z (or $\theta = 0$) direction. The surface analog of Eq. (1.20) gives:

$$E_z(0) \;=\; \int \frac{(\rho_s)_b}{a^2}(-\mathbf{e}_r\cdot\mathbf{e}_z)\,2\pi a^2\sin\theta\,d\theta$$

$$= -\,2\pi P_0\int_0^{\pi}\cos^2\theta\,\sin\theta\,d\theta \;=\; -\,2\pi P_0\int_{-1}^{+1}u^2\,du \;=\; -\frac{4\pi}{3}P_0$$

That is, $\mathbf{E}(0) = -(4\pi/3)\mathbf{P}_0$.

(b) The total dipole moment of the polarized sphere is $p = P_0(4\pi a^3/3)$, where a is the radius. Since the external field of a spherical charge distribution is the same as an equivalent point charge at the origin (Prob. 1-4), the dipole moment of the superposed uniformly charged spheres is $p = = q\delta = \rho_0(4\pi a^3/3)\delta$. These are equal (independent of a) when

$$P_0 \;=\; \rho_0\delta$$

Now if we take the origin at the center of the negative sphere, with the center of the positive sphere at the vector position $\boldsymbol{\delta}$, we can write the total field within the superposed spheres as

$$\mathbf{E} \;=\; \frac{4\pi}{3}\rho_0\,[-\,\mathbf{r} + (\mathbf{r}-\boldsymbol{\delta})] \;=\; -\frac{4\pi}{3}\rho_0\boldsymbol{\delta} \;=\; -\frac{4\pi}{3}\mathbf{P}$$

Thus the result of Part (a) extends to the *entire volume* of the polarized sphere. [Yet another approach to this problem uses the spherical harmonic expansion of Section 3.3.]

(c) Again we use superposition. If there were no cavity, the (spatial average) field in the uniform dielectric would be **E**, as given. Superpose the polarized sphere of Parts (a) and (b) with its polarization equal and opposite to the polarization in the dielectric, producing the cavity, in which the net polarization is zero and the field is

$$\mathbf{E} - \frac{4\pi}{3}(-\mathbf{P}) \;=\; \mathbf{E} + \frac{4\pi}{3}\mathbf{P}$$

$$\Phi_m(r > a) = \sum_l C_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

where we have renamed the "B_l" coefficients in Eq. (3.39) to avoid confusion.

We must now investigate the boundary conditions that hold at $r = a$. From Eqs. (1.92) and (1.94), we conclude that the radial (normal) component of **B** must be continuous,

$$B_r(\text{inside}) = B_r(\text{outside})$$

and that the θ (tangential) components differ by $4\pi K/c$ according to

$$B_\theta(\text{outside}) - B_\theta(\text{inside}) = \frac{4\pi}{c} K_\varphi$$

These boundary conditions couple the inside and outside regions together, and allow us to make the important inference that, because of the orthogonality of the Legendre polynomials, only the term involving $P_1 = \cos\theta$ is acceptable for $\Phi_m(\text{outside})$. Accordingly, we can write the two boundary conditions explicitly as

$$B_r = -\frac{\partial \Phi_m}{\partial r} \quad \Longrightarrow \quad B_0 \cos\theta = \frac{2C_1}{a^3} \cos\theta$$

$$B_\theta = -\frac{\partial \Phi_m}{r\,\partial \theta} \quad \Longrightarrow \quad \frac{C_1}{a^3} \sin\theta + B_0 \sin\theta = \frac{4\pi}{c} K_\varphi(\theta)$$

These simultaneous equations can be solved for

$$C_1 = \frac{a^3 B_0}{2}; \qquad K_\varphi(\theta) = \frac{3c}{8\pi} B_0 \sin\theta$$

The current density varies as $\sin\theta$, flowing azimuthally on the surface of the sphere. Translated into discrete turns of wire, this is equivalent to a coil with *constant axial pitch*. The external field of the coil is that of a pure dipole.

This problem is closely related to Probs. 1-13 and 2-24. For instance, the magnetic analog of Prob. 1-13, which showed that the internal field of a uniformly polarized sphere is constant, gives the internal field of a uniformly magnetized sphere as

$$\mathbf{H} = -\frac{4\pi}{3} \mathbf{M}$$

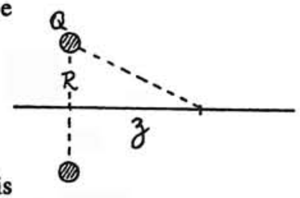
from which

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = +\frac{8\pi}{3} \mathbf{M}$$

Then from Eq. (1.69),

$$\mathbf{K} = -c\,\mathbf{n} \times \mathbf{M} \;\rightarrow\; \frac{3c}{8\pi} B_0 \sin\theta\, \mathbf{e}_\varphi$$

3-26. The potential on the symmetry axis can be written down immediately:



$$\Phi(z) = \frac{Q}{(z^2 + R^2)^{1/2}}$$

For regions where z is less than or greater than R, this function can be expanded in the respective power series:

$$\left.\begin{aligned} \Phi_1(z < R) &= \frac{Q}{R}\left[1 - \frac{1}{2}\left(\frac{z}{R}\right)^2 + \frac{3}{8}\left(\frac{z}{R}\right)^4 + \cdots\right] \\ \Phi_2(z > R) &= \frac{Q}{z}\left[1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + \frac{3}{8}\left(\frac{R}{z}\right)^4 + \cdots\right] \end{aligned}\right\} \quad (1)$$

Now imagine a spherical surface of radius R dividing all space into two regions. The inner region, $r < R$, contains no charge (we neglect the thickness of the ring, which lies on the surface $r = R$). Therefore the potential in this region is a solution of Laplace's equation and can be written in the form

$$\Phi_1(r < R) = A_0 + A_1 \left(\frac{r}{R}\right) P_1 + A_2 \left(\frac{r}{R}\right)^2 P_2 + \cdots$$

Similarly for $r > R$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big\}\quad (2)$

$$\Phi_2(r > R) = A_0 \left(\frac{R}{r}\right) + A_1 \left(\frac{R}{r}\right)^2 P_1 + A_2 \left(\frac{R}{r}\right)^3 P_2 + \cdots$$

Note that we have labeled the terms so that the two series coincide at $r = R$. On the z (polar) axis, $r \to z$, $\theta \to 0$, and these series reduce to:

$$\left.\begin{aligned} \Phi_1 &= A_0 + A_1 \left(\frac{z}{R}\right) + A_2 \left(\frac{z}{R}\right)^2 + \cdots \\ \Phi_2 &= A_0 \left(\frac{R}{z}\right) + A_1 \left(\frac{R}{z}\right)^2 + A_2 \left(\frac{R}{z}\right)^3 + \cdots \end{aligned}\right\} \quad (3)$$

Comparing Eqs. (3) with (1), we can evaluate the coefficients as:

8-12. If we neglect the radiation energy loss, then u is related to r along the trajectory by

$$\tfrac{1}{2}m u_0^2 = \tfrac{1}{2}mu^2 + \frac{Ze^2}{r}$$

As a perturbation, the energy lost to radiation is given by the nonrelativistic Larmor formula, Eq. (8.89), as

$$\Delta W = \int_0^\infty P(t)\,dt = \frac{2e^2}{3c^3}\int \frac{a^2}{u}\,dr$$

But,

$$a = -\frac{1}{m}\frac{dU}{dr} = \frac{Ze^2}{mr^2}$$

$$u = \mp\sqrt{u_0^2 - \frac{2Ze^2}{mr}}$$

Because of the reentrant path, we calculate the integral during the outbound segment from $r_{\min} = 2Ze^2/mu_0^2$ to $r \to \infty$, and double the result:

$$\Delta W = \frac{2e^2}{3c^3}\left(\frac{Ze^2}{m}\right)^2 \frac{2}{u_0}\int_{r_{\min}}^\infty \frac{dr}{r^4\left(1 - \frac{r_{\min}}{r}\right)^{1/2}}$$

Let $x = 1/r$, and $dr = -dx/x^2$, to obtain the integral

$$\int_{r_{\min}}^\infty \frac{dr}{\cdots} = \int_0^{x_{\max}} \frac{x^2\,dx}{\left(1 - \frac{x}{x_{\max}}\right)^{1/2}} = \left[-2x_{\max}^3\left(\frac{X^{5/2}}{5} - \frac{2X^{3/2}}{3} + X^{1/2}\right)\right]_0^{x_{\max}}$$

$$= \frac{16}{15}x_{\max}^3 = \frac{16}{15}\left(\frac{mu_0^2}{2Ze^2}\right)^3$$

(where $X = 1 - x/x_{\max}$, as in Dwight §191.21). Thus,

$$\Delta W = \frac{2e^2}{3c^3}\left(\frac{Ze^2}{m}\right)^2 \frac{2}{u_0}\,\frac{16}{15}\left(\frac{mu_0^2}{2Ze^2}\right)^3 = \frac{8mu_0^5}{45Zc^3} = \frac{16}{45}\,\frac{\beta_0^3}{Z}\left(\tfrac{1}{2}mu_0^2\right)$$

Since by hypothesis $\beta_0 = u_0/c \ll 1$, the perturbation approach is justified.

8-13. The geometry is defined in Fig. 8-7, and the radiation pattern is given by Eq. (8.106). **(a)** For the orbital plane, the azimuthal angle φ is 0 or π, and θ measures the polar angle from **u**. Inspection of Eq. (8.106) shows that the dependence on φ and θ is such that we can suppress the double-valued φ and interpret θ as an *azimuthal* angle in this plane (i.e., $0 \le \theta \le 2\pi$). Figure 8-9 plots the magnitude of $dP/d\Omega$ as a function of θ with **u** (to the right) and **a** (up or down). The pattern is symmetrical about the **u** axis, and the numerator simplifies to

$$\begin{aligned}
&(1 - \beta\cos\theta)^2 - (1 - \beta^2)\sin^2\theta \\
&\quad = (1 - 2\beta\cos\theta + \beta^2\cos^2\theta) - (1 - \beta^2)(1 - \cos^2\theta) \\
&\quad = -2\beta\cos\theta + \beta^2 + \cos^2\theta \\
&\quad = (\cos\theta - \beta)^2
\end{aligned}$$

Thus in the orbital plane the angular dependence reduces to

$$\left(\frac{dP}{d\Omega}\right)_{u\text{-}a\ \text{plane}} \propto \frac{(\cos\theta - \beta)^2}{(1 - \beta\cos\theta)^5} \qquad (1)$$

The nulls occur when $\cos\theta = \beta$, and for the examples of Fig. 8-9:

$\cos^{-1}(0.1) = 84.26°$ $\qquad$ $\cos^{-1}(0.7) = 45.57°$
$\cos^{-1}(0.3) = 72.54°$ $\qquad$ $\cos^{-1}(0.9) = 25.84°$
$\cos^{-1}(0.5) = 60°$

(b) The maximum value of (1) [and of the full Eq. (8.106)] occurs for $\theta = 0$, for which $(dP/d\Omega)_{\max} = 1/(1-\beta)^3$. Let α be the value of θ at which the intensity falls to one-half of maximum, so that the beamwidth as defined is $\Delta\theta \equiv 2\alpha$. Expand the cosine for small angles ($\cos\alpha \to 1 - \frac{1}{2}\alpha^2$) to write the half-power condition as:

$$\frac{(1 - \frac{1}{2}\alpha^2 - \beta)^2}{(1 - \beta + \frac{1}{2}\beta\alpha^2)^5} = \frac{1}{2(1 - \beta)^3}$$

Cross-multiplying and discarding terms beyond α^2, we have

$$2(1 - \beta)^3\left[(1 - \beta)^2 - 2(1 - \beta)(\tfrac{1}{2}\alpha^2)\right] \approx (1 - \beta)^5 + 5(1 - \beta)^4(\tfrac{1}{2}\beta\alpha^2)$$

$$(1 - \beta)^5 \approx (1 - \beta)^4\,(2\alpha^2 + \tfrac{5}{2}\beta\alpha^2)$$

$$(\Delta\theta)_{u\text{-}a\ \text{plane}} = 2\alpha \approx 2\sqrt{\frac{1 - \beta}{2 + \frac{5}{2}\beta}}$$

$$\to \frac{2\sqrt{2}}{3}\sqrt{1 - \beta} = 0.943\sqrt{1 - \beta}$$