Solutions Manual For

CONTINUUM ELECTROMECHANICS

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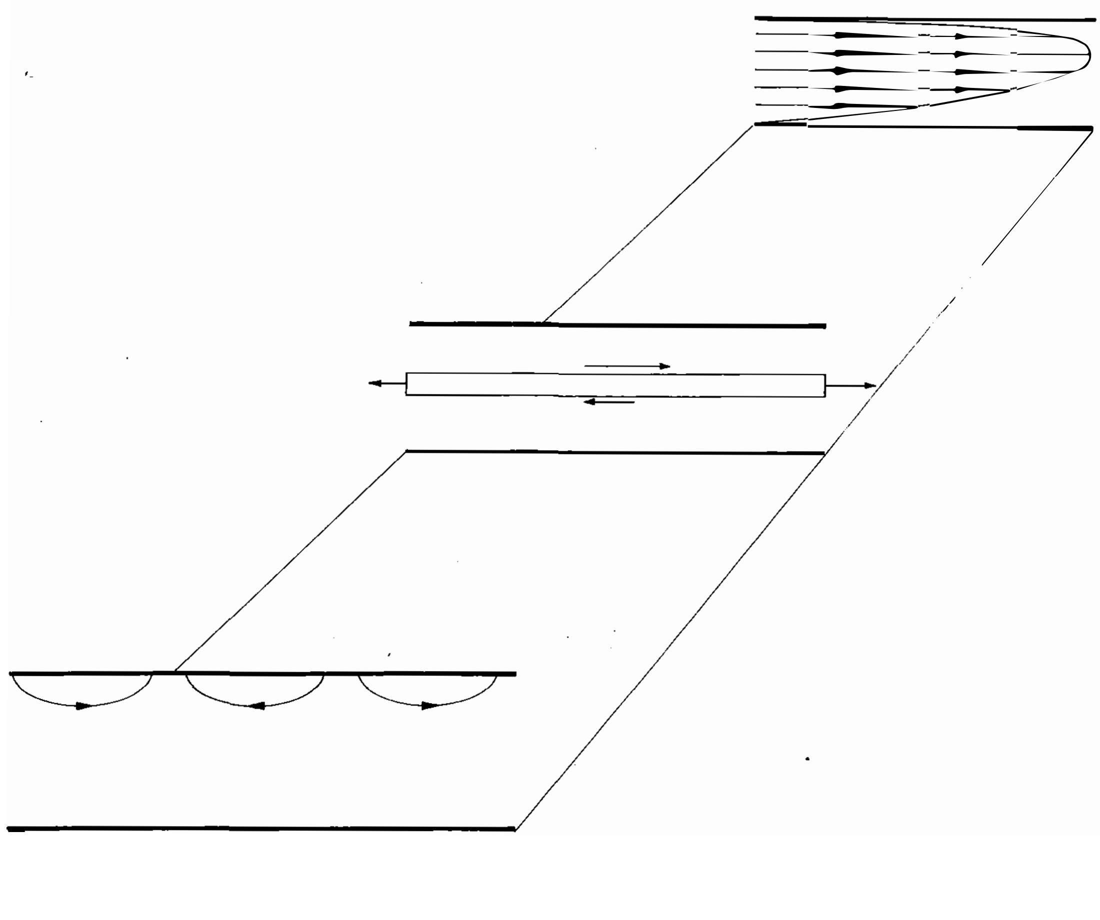

Prob. 2.3.1(cont.)

Note that the surface charge on the lower electrode, as well as the surface current density there, are related to the fields between the electrodes by

$$\sigma_f = E_x \quad ; \quad K_z = H_y \tag{21}$$

The respective quantities on the upper electrode are the negatives of these quantities.(Gauss'law and Ampere's law). With Eqs. 7 used to recover the time dependence, what have been found to second order in β are the normalized fields

$$E_x = z\left[1 - \tfrac{1}{2}\left(\tfrac{1}{3}z^2 - 1\right)\beta + \tfrac{1}{4}\left(\tfrac{1}{30}z^4 - \tfrac{1}{3}z^2 + \tfrac{5}{6}\right)\beta^2\right]\sin t = \sigma_f \tag{22}$$

$$H_y = \left[1 - \tfrac{1}{2}(z^2 - 1)\beta + \tfrac{1}{4}\left(\tfrac{1}{6}z^4 - z^2 + \tfrac{5}{6}\right)\beta^2\right]\cos t = K_z \tag{23}$$

The dimensioned forms follow by identifying

$$E_x = \frac{\mu_o \omega l I_o}{w} \tag{24}$$

e) Now, consider the exact solutions. Eqs. 7 substituted into Eqs. 5 and 6 give

$$\frac{d^2\hat{H}_y}{dz^2} + \beta\hat{H}_y = 0 \tag{25}$$

$$\hat{E}_x = \frac{j}{\beta}\frac{d\hat{H}_y}{dz} \tag{26}$$

Solutions that satisfy these expressions as well as Eqs. 13 are

$$\hat{H}_y = \cos(\sqrt{\beta}\,z)/\cos\sqrt{\beta} \tag{27}$$

$$\hat{E}_x = \frac{-j}{\sqrt{\beta}}\sin(\sqrt{\beta}\,z)/\cos\sqrt{\beta} \tag{28}$$

These can be expanded to second order in β as follows.

$$\hat{H}_y \cong \frac{1 - \frac{1}{2}\beta z^2 + \frac{1}{4!}\beta^2 z^4 + \cdots}{1 - \frac{1}{2}\beta + \frac{1}{4!}\beta^2 + \cdots} \tag{29}$$

$$\cong \left(1 - \tfrac{1}{2}\beta z^2 + \tfrac{1}{4!}\beta^2 z^4\right)\left(1 - \left(-\tfrac{1}{2}\beta + \tfrac{1}{4!}\beta^2\right) + \left(-\tfrac{1}{2}\beta + \tfrac{1}{4!}\beta^2\right)^2\right)$$

$$= \left(1 - \tfrac{1}{2}\beta z^2 + \tfrac{1}{24}\beta^2 z^4\right)\left(1 + \tfrac{1}{2}\beta + \tfrac{5}{24}\beta^2 - \tfrac{1}{24}\beta^3 + \tfrac{1}{576}\beta^4\right)$$

$$\cong 1 - \tfrac{1}{2}(z^2 - 1)\beta + \tfrac{1}{4}\left(\tfrac{1}{6}z^4 - z^2 + \tfrac{5}{6}\right)\beta^2$$

Prob. 2.16.1 (cont.)

The second form, Eq. (b), is obtained by applying Cramer's rule to the inversion of Eq. 8. Note that the determinant of the coefficients is

$$\text{Det} = -\coth^2\gamma\Delta + \frac{1}{\sinh^2\gamma\Delta} = \frac{1-\cosh^2\gamma\Delta}{\sinh^2\gamma\Delta} = -1 \qquad (7)$$

so

$$\begin{bmatrix}\tilde{\Phi}^\alpha \\ \\ \tilde{\Phi}^\beta\end{bmatrix} = \frac{1}{\epsilon\gamma}\begin{bmatrix} -\coth\gamma\Delta & \dfrac{1}{\sinh\gamma\Delta} \\ \\ \dfrac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta\end{bmatrix}\begin{bmatrix}\tilde{D}_x^\alpha \\ \\ \tilde{D}_x^\beta\end{bmatrix} \qquad (8)$$

Prob. 2.16.2 For the limit m=0,k=0, solutions are combined to satisfy the potential constraints by Eq. 2.16.20, and it follows that the electric displacement is

$$\tilde{D}_r = -\epsilon\frac{\partial\tilde{\Phi}}{\partial r} = -\epsilon\,\tilde{\Phi}^\alpha\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} + \epsilon\,\tilde{\Phi}^\beta\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} \qquad (1)$$

This is evaluated at the respective boundaries to give Eq. (a) of Table 2.16.2 with f_m and g_m as defined for k=0,m=0.

For k=0,m≠0, the correct combination of potentials is given by Eq. 2.16.21. It follows that

$$\tilde{D}_r = \epsilon m\left\{\frac{\tilde{\Phi}^\alpha}{\beta}\left[\frac{\left(\frac{\beta}{r}\right)^{m+1}+\left(\frac{r}{\beta}\right)^{m-1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right] - \frac{\tilde{\Phi}^\alpha}{\alpha}\left[\frac{\left(\frac{r}{\alpha}\right)^{m-1}+\left(\frac{\alpha}{r}\right)^{m+1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right]\right\} \qquad (2)$$

Evaluation of this expression at the respective boundaries gives Eqs. (a) of Table 2.16.2 with entries f_m and g_m as defined for the case k=0,m=0.

For k ≠ 0,m ≠ 0, the potential is given by Eq. 2.16.25. Thus, the electric displacement is

$$\tilde{D}_r = -jk\left\{\tilde{\Phi}^\alpha\frac{[H_m(jk\beta)J_m'(jkr) - J_m(jk\beta)H_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right.$$
$$\left. + \tilde{\Phi}^\beta\frac{[J_m(jk\alpha)H_m'(jkr) - H_m(jk\alpha)J_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right\} \qquad (3)$$

and evaluation at the respective boundaries gives Eqs. (a) of the table with f_m and g_m as defined in terms of H_m and J_m. To obtain g_m in the form given, Prob. 3.10.6

Showing that the identity holds is a matter of simply writing out the components in cartesian coordinates. The i'th component of the force density is then written using the identity to write $\bar{J}\times\bar{B}$ where $\bar{J} = \nabla\times\bar{H}$.

$$F_i = \frac{\partial H_i}{\partial x_j}B_j - \frac{\partial H_j}{\partial x_i}B_j + \sum_{k=1}^{m}\frac{\partial W}{\partial \alpha_k}\frac{\partial \alpha_k}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right) \qquad (1)$$

In the first term, B_j is moved inside the derivative and the condition $\partial B_j/\partial x_j = \nabla\cdot\bar{B} = 0$ exploited. The third term is replaced by the magnetic analogue of Eq. 3.7.26.

$$F_i = \frac{\partial}{\partial x_j}(H_iB_j) - \frac{\partial H_j}{\partial x_i}B_j + B_j\frac{\partial H_j}{\partial x_i} - \frac{\partial}{\partial x_i}(B_jH_j) + \frac{\partial W}{\partial x_i} - \frac{\partial}{\partial x_i}\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k} \qquad (2)$$

The second and third terms cancel, so that this expression can be rewritten

$$F_i = \frac{\partial}{\partial x_j}\left[H_iB_j - \delta_{ij}\left(W' + \sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right)\right];\ W' \equiv \bar{B}\cdot\bar{H} - W \qquad (3)$$

and the stress tensor identified as the quantity in brackets.

Problem 3.10.7 The i'th component of the force density is written using the identity of Prob. 2.10.5 to express $\bar{J}_f \times \mu_o\bar{H} = (\nabla\times\bar{H})\times\mu_o\bar{H}$

$$F_i = \mu_o\left(\frac{\partial H_i}{\partial x_j}H_j\right) - \mu_o\frac{\partial H_j}{\partial x_i}H_j + \mu_o M_j\frac{\partial H_i}{\partial x_j} \qquad (1)$$

This expression becomes

$$F_i = \frac{\partial}{\partial x_j}(\mu_o H_iH_j) - H_i\frac{\partial}{\partial x_j}(\mu_o H_j) - \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\mu_o H_jH_j\right) + \frac{\partial}{\partial x_j}(\mu_o M_jH_i) - H_i\frac{\partial}{\partial x_j}(\mu_o M_j) \qquad (2)$$

where the first two terms result from the first term in F_i, the third term results from taking the H_j inside the derivative and the last two terms are an expansion of the last term in F_i. The second and last term combine to give $\nabla\cdot\mu_o(\bar{H}+\bar{M}) \equiv \nabla\cdot\bar{B} = 0$. Thus, with $\bar{B} = \mu_o(\bar{H}+\bar{M})$, the expression takes the proper form for identifying the stress tensor.

$$F_i = \frac{\partial}{\partial x_j}\left[\mu_o(M_j + H_j)H_i - \delta_{ij}\tfrac{1}{2}\mu_o H_kH_k\right] \qquad (3)$$

Prob. 7.19.2 The laws required to represent the elastic displacements and stresses are given in Table P7.16.1. In terms of $\bar{A}_s$ and ψ_s as defined in Prob. 7.18.1, Eq. (e) becomes

$$\nabla\times\left[\rho\frac{\partial^2\bar{A}_s}{\partial t^2}+G_s\nabla\times\nabla\times\bar{A}_s\right]-\nabla\left[\rho\frac{\partial^2\psi_s}{\partial t^2}-(2G_s+\lambda_s)\nabla^2\psi_s\right]=0 \tag{1}$$

Given that because $\nabla\cdot\bar{A}_s=0$, $\nabla\times\nabla\times\bar{A}_s=-\nabla^2\bar{A}_s$, this expression is satisfied if

$$\frac{\partial^2\bar{A}_s}{\partial t^2}=\frac{G_s}{\rho}\nabla^2\bar{A}_s \Rightarrow \frac{\partial^2\bar{A}}{\partial t^2}=\frac{G_s}{\rho}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial^2\psi_s}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\nabla^2\psi_s \Rightarrow \frac{\partial^2\psi}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) \tag{3}$$

In the second equations, $\bar{A}_s=A(x,y,t)\bar{i}_z$, $\psi_s=\psi(x,y,t)$, to represent the two-dimensional motions of interest.

Given solutions to Eqs. 2 and 3, $\bar{\xi}$ is evaluated.

$$\xi_x=\frac{\partial A}{\partial y}-\frac{\partial\psi}{\partial x} \tag{4}$$

$$\xi_y=-\frac{\partial A}{\partial x}-\frac{\partial\psi}{\partial y} \tag{5}$$

The desired stress components then follow from Eqs. (a) and (b) from Table P7.16.1.

$$S_{xx}=(2G_s+\lambda_s)\frac{\partial\xi_x}{\partial x}+\lambda_s\frac{\partial\xi_y}{\partial y} \tag{6}$$

$$S_{yx}=G_s\left(\frac{\partial\xi_y}{\partial x}+\frac{\partial\xi_x}{\partial y}\right) \tag{7}$$

In particular, solutions of the form $A=\mathrm{Re}\,\hat{A}(x)e^{j(\omega t-ky)}$ and

Prob. 9.18.2 Because the channel is designed to make the temperature constant, it follows from the mechanical equation of state (Eq. 9.18.13) that

$$p = \rho R T \Rightarrow \frac{p}{p_d} = \frac{\rho}{\rho_d} \tag{1}$$

At the same time, it has been shown that the transition is adiabatic, so Eq. 9.18.23 holds.

$$\frac{p}{p_d} = \left(\frac{\rho}{\rho_d}\right)^{\gamma} ; \; \gamma \neq 1 \tag{2}$$

Thus, it follows that both the temperature and mass density must also be constant

$$p = p_d \; ; \; \rho = \rho_d \tag{3}$$

In turn, Eq. 9.18.10, which expresses mass conservation, becomes

$$v A = v_d A_d \tag{4}$$

and Eq. 9.18.20 can be used to show that the charge density is constant

$$\rho_f = \frac{I}{A_d v_d} \tag{5}$$

So, with the relation $E = -d\Phi/dz$, Eq. 9.18.9 is (Gauss' Law)

$$-\frac{d}{dz}\left(A\frac{d\Phi}{dz}\right) = \frac{A}{\epsilon_o}\frac{I}{A_d v_d} \tag{6}$$

In view of the isothermal condition, Eq. 9.18.22 requires that

$$\frac{1}{2}v^2 + \frac{I}{\rho_d A_d v_d}\Phi = \frac{1}{2}v_d^2 + \frac{I}{\rho_d A_d v_d}\Phi_d \tag{7}$$

The required relation of the velocity to the area is gotten from Eq. 3.

$$v = v_d \frac{A_d}{A} \tag{8}$$

and substitution of this relation into Eq. 7 gives the required expression for Φ in terms of the area.

$$\Phi = \frac{1}{2}v_d^2\left(1 - \frac{A_d^2}{A^2}\right)\frac{\rho_d A_d v_d}{I} + \Phi_d \tag{9}$$

Substitution of this expression into Eq. 5 gives the differential equation for the area dependence on z that must be used to secure a constant temperature.

$$\frac{d}{dz^2}A^{-1} - k^2 A = 0 \; ; \; k^2 \equiv \frac{\rho_{fd} I \rho_d^2}{\epsilon_o (A_d \rho_d v_d)^3} \tag{10}$$

Prob. 11.12.2 The dispersion equation, without the long-wave approximation, is given by Eq. 8. Solved for ω it gives one root

$$\omega = k + \frac{jk}{U}\tanh k \tag{1}$$

That is, there is only one temporal mode and it is stable. This is sufficient condition to identify all spatial modes as evanescent.

The long-wave limit, if represented by Eq. 11, is not self-consistent. This is evident from the fact that the expression is quadratic in ω and it is clear that an extraneous root has been introduced by the polynomial approximation to the transcendental functions. In fact, two higher order terms must be omitted to make the -k relation self-consistent, and Eq. 5.7.11 becomes

$$k_{\mp} = j\frac{U}{2} \pm \sqrt{-\frac{U^2}{4} - j\omega U} \tag{2}$$

Solved for ω, this expression gives

$$\omega = k\left(1 + \frac{jk}{U}\right) \tag{3}$$

which is directly evident from Eq. 1.

To plot the loci of k for fixed values of ω_r as σ goes from ∞ to zero, Eq. 2 is written as

$$k_{\mp} = j\left[\frac{U}{2} \pm \sqrt{\left(\frac{U^2}{4} + \sigma U\right) + j\omega_r U}\right] \tag{4}$$

The loci of k are illustrated by the figure with $\underline{U} = 0.2$.