Instructor's Solutions Manual

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Differential Equations With Boundary Value Problems

SECOND EDITION

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that $y_1(t) = -1$ and $y_2(t) = 1$ are both solutions to the differential equation. If y is a solution and satisfies $y(1) = 0$, then $y_1(1) < y(1) < y_2(1)$. By the uniqueness theorem we must have $y_1(t) < y(t) < y_2(t)$ for all t for which y is defined. Hence $-1 < y(t) < 1$ for all t for which y is defined.

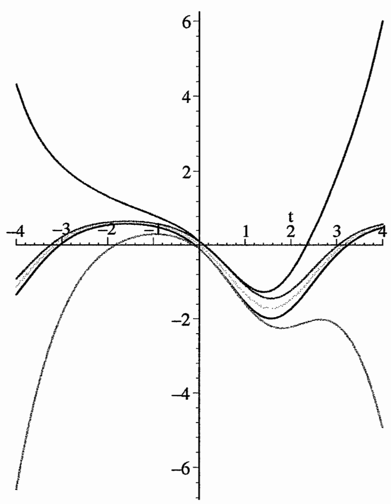
30. Notice that $x_1(t) = 0$ and $x_2(t) = 1$ are solutions to the same differential equation with initial values $x_1(0) = 0 < 1/2 = x(0) < 1 = x_2(0)$. The right hand side of the differential equation, $f(t, x) = (x^3 - x)/(1 + t^2x^2)$, and
$$\frac{\partial f}{\partial x} = \frac{(3x^2 - 1)(1 + t^2x^2) - 2t^2x(x^3 - x)}{(1 + t^2x^2)^2},$$
are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x, x_1, and x_2 cannot cross. Hence we must have $0 = x_1(t) < x(t) < x_2(t) = 1$ for all t.

31. Notice that $x_1(t) = t^2$ is a solution to the same differential equation with initial value $x_1(0) = 0 < 1 = x(0)$. The right hand side of the differential equation, $f(t, x) = x - t^2 + 2t$ and $\partial f/\partial x = 1$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x and x_1 cannot cross. Hence we must have $t^2 = x_1(t) < x(t)$ for all t.

32. Notice that $y_1(t) = \cos t$ is a solution to the same differential equation with initial value $y_1(0) = 1 < 2 = y(0)$. The right hand side of the differential equation, $f(t, y) = y^2 - \cos^2 t - \sin t$ and $\partial f/\partial y = 2y$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for y and y_1 cannot cross. Hence we must have $y(t) > y_1(t) = \cos t$ for all t.

Section 2.8. Dependence of Solutions on Initial Conditions

1. $x(0) = 0.8009$
2. $x(0) = .9084$
3. $x(0) = 0.9596$
4. $x(0) = 0.9826$
5. $x(0) = 0.7275$
6. $x(0) = 0.72897$
7. $x(0) = 0.7290106$
8. $x(0) = 0.729011125$
9. $x(0) = -3.2314$
10. $x(0) = -3.23208$
11. $x(0) = -3.2320923$
12. $x(0) = -3.23092999999$
13. Ten! :-)
14. $1 - e^{\sin t} - (1/10)e^{|t|} \le y(t) \le 1 - e^{\sin t} + (1/10)e^{|t|}$



Chapter 4. Second-Order Equations

Section 4.1. Definitions and Examples

1. Compare

$$y'' + 3y' + 5y = 3\cos 2t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 3$, $q(t) = 5$, and $g(t) = 3\cos 2t$. Hence, the equation is linear and inhomogeneous.

2. Divide both sides of $t^2y'' = 4y' - \sin t$ by t^2, then rearrange to obtain

$$y'' - \frac{4}{t^2}y' = -\frac{\sin t}{t^2}.$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = -4/t^2$, $q(t) = 0$, and $g(t) = -(\sin t)/t^2$. Hence, the equation is linear and inhomogeneous.

3. Expand $t^2y'' + (1 - y)y' = \cos 2t$ to obtain

$$t^2y'' + y' - yy' = \cos 2t.$$

Note that the term yy' is nonlinear. Hence, this equation is nonlinear.

4. Divide both sides of $ty'' + (\sin t)y' = 4y - \cos 5t$ by t, then rearrange to obtain

$$y'' + \frac{\sin t}{t}y' - \frac{4}{t} = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = (\sin t)/t$, $q(t) = -4/t$, and $g(t) = -(\cos 5t)/t$. Hence, the equation is linear and inhomogeneous.

5. In

$$t^2y'' + 4yy' = 0,$$

note that the term $4yy'$ is nonlinear. Hence, this equation is nonlinear.

6. Compare

$$y'' + 4y' + 7y = 3e^{-t}\sin t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 4$, $q(t) = 7$, and $g(t) = 3e^{-t}\sin t$. Hence, the equation is linear and inhomogeneous.

7. In

$$y'' + 3y' + 4\sin y = 0$$

note that the term $4\sin y$ is nonlinear. Hence, this equation is nonlinear.

8. Divide both sides of $(1 - t^2)y'' = 3y$ by $1 - t^2$, then rearrange the terms to obtain

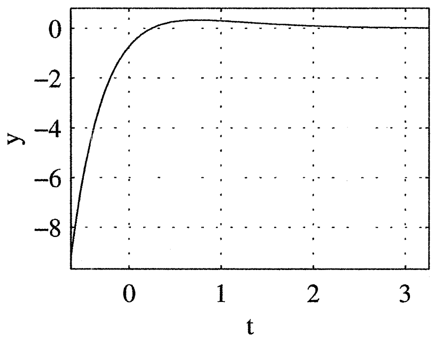
$$y'' - \frac{3}{1 - t^2}y = 0.$$

Compare with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 0$, $q(t) = -3/(1 - t^2)$, and $g(t) = 0$. Hence, the equation is linear and homogeneous.

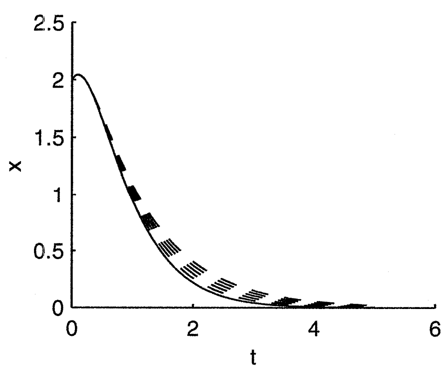
t.



Note that the graph crosses the t-axis exactly once. Finally, by picking initial conditions from the unshaded region, you will note that this solution also crosses the y-axis exactly once, but at $t < 0$.

20. (a) The system $x'' + \mu x' + 4x = 0$ has characteristic equation $\lambda^2 + \mu\lambda + 4 = 0$. If $\mu = 4$, this becomes $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$, and there is one repeated root, hence the critically damped case.

(b) The critically damped solution (solid line in figure) approaches the t-axis faster than any of the other overdamped solutions.



An overdamped screen door will shut on its own without slamming. A critically damped door will shut as fast as possible without slamming.

21. (a) Suppose that $mx'' + \mu x' + kx = 0$ is overdamped. We can write

$$x'' + \frac{\mu}{m}x' + \frac{k}{m}x = 0$$
$$x'' + 2cx' + \omega_0^2 = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The system has characteristic equation $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and zeros

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and}$$
$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

If w4e set $\gamma = \sqrt{c^2 - \omega_0^2}$, then

$$\lambda_1 = -c - \gamma \quad \text{and} \quad \lambda_2 = -c + \gamma,$$

and $\lambda_2 - \lambda_1 = 2\gamma$. If the system is overdamped, note that

$$\begin{aligned} c^2 - \omega_0^2 &> 0 \\ \left(\frac{\mu}{2m}\right)^2 &> \frac{k}{m} \\ \mu^2 &> 4mk \\ \mu &> 2\sqrt{mk}. \end{aligned}$$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

The initial condition $x(0) = 0$ gives $0 = C_1 + C_2$ and $C_1 = -C_2$. Differentiating $x(t)$,

$$x'(t) = C_1\lambda_1 e^{\lambda_1 t} + C_2\lambda_2 e^{\lambda_2 t},$$

and the initial condition $x'(0) = v_0$ provides $v_0 = C_1\lambda_1 + C_2\lambda_2$. This system is easily solved for

$$C_1 = \frac{v_0}{\lambda_1 - \lambda_2} = -\frac{v_0}{2\gamma} \quad \text{and}$$
$$C_2 = \frac{-v_0}{\lambda_1 - \lambda_2} = \frac{v_0}{2\gamma},$$

$C_1 = -3/5$ and $C_2 = -11/20$. Therefore, the solution is

$$y = e^t\left(-\frac{3}{5}\cos 2t - \frac{11}{20}\sin 2t\right) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

22. The homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ and repeated root $\lambda = -2$. Thus the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{-2t}.$$

The particular solution $y_p = at + b$ has derivatives $y_p' = a$ and $y_p'' = 0$, which when substituted in $y'' + 4y' + 4y = 4 - t$,

$$\begin{aligned} 4a + 4(at + b) &= 4 - t \\ 4at + (4a + 4b) &= -t + 4. \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 4a &= -1 \\ 4a + 4b &= 4, \end{aligned}$$

which has solution $a = -1/4$ and $b = 5/4$. Thus, the general solution is

$$y = (C_1 + C_2 t)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

The initial condition $y(0) = -1$ provides

$$-1 = C_1 + \frac{5}{4}.$$

Differentiate.

$$y' = C_2 e^{-2t} - 2e^{-2t}(C_1 + C_2 t) - \frac{1}{4}$$

The initial condition $y'(0) = 0$ provides

$$0 = C_1 - 2C_1 - \frac{1}{4}.$$

Thus, $C_1 = -9/4$ and $C_2 = -17/4$ and

$$y = \left(-\frac{9}{4} - \frac{17}{4}t\right)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

23. The homogeneous equation $y'' - 2y' + y = 0$ has characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, with repeated zero $\lambda = 1$. Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^t.$$

The particular solution $y_p = at^3 + bt^2 + ct + d$ has derivatives

$$\begin{aligned} y_p' &= 3at^2 + 2bt + c \\ y'' &= 6at + 2b, \end{aligned}$$

which when substituted in $y'' - 2y' + y = t^3$, rearranging, yields

$$at^3 + (-6a + b)t^2 + (6a - 4b + c)t + (2b - 2c + d) = t^3.$$

Thus,

$$\begin{aligned} a &= 1 \\ -6a + b &= 0 \\ 6a - 4b + c &= 0 \\ 2b - 2c + d &= 0, \end{aligned}$$

which has solution $a = 1$, $b = 6$, $c = 18$, and $d = 24$. Thus, the general solution is

$$y = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24.$$

The initial condition $y(0) = 1$ gives $1 = C_1 + 24$. Differentiating,

$$y' = C_2 e^t + (C_1 + C_2 t)e^t + 3t^2 + 12t + 18.$$

The initial condition $y'(0) = 0$ gives $0 = C_2 + C_1 + 18$. The system has solution $C_1 = -23$ and $C_2 = 5$. Therefore, the solution is

$$y = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24.$$

24. The homogeneous equation $y'' - 3y' - 10y = 0$ has characteristic equation $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$ with zeros $\lambda_1 = 5$ and $\lambda_2 = -2$. Thus, the homogeneous solution is

$$y_h = C_1 e^{5t} + C_2 e^{-2t}.$$

Thus, the forcing term of $y'' - 3y' - 10y = 3e^{-2t}$ is a solution of the homogeneous equation. Substitute $y_p = Ate^{-2t}$ and its derivatives

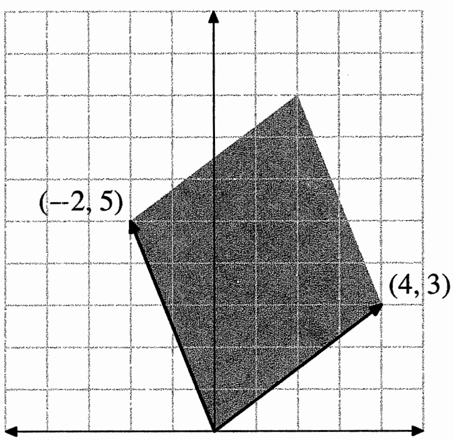
$$\begin{aligned} y_p' &= Ae^{-2t}(1 - 2t) \\ y_p'' &= (-4 - 4t)Ae^{-2t} \end{aligned}$$

The determinant is

$$\begin{aligned}|\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 1 & 6 \\ 4 & 1 \end{vmatrix} = (1)(1) - (4)(6) \\ &= 1 - 24 = -23.\end{aligned}$$

Note that the determinant is the negative of the area.

4. Estimate the area by counting square units inside the parallelogram in

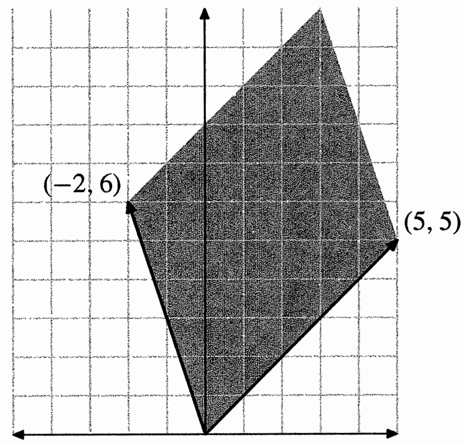


The determinant is

$$\begin{aligned}|\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = (-2)(3) - (5)(4) \\ &= -6 - 20 = -26.\end{aligned}$$

Note that the determinant is the negative of the area.

5. Estimate the area by counting square units inside the parallelogram in



The determinant is

$$\begin{aligned}|\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 5 & -2 \\ 5 & 6 \end{vmatrix} = (5)(6) - (5)(-2) \\ &= 30 + 10 = 40.\end{aligned}$$

6. First note the determinant of A.

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To prove part (1), construct B by adding r times row 1 to row 2.

$$B = \begin{pmatrix} a & b \\ c + ra & d + rb \end{pmatrix}.$$

Then,

$$\begin{aligned}|B| &= \begin{vmatrix} a & b \\ c + ra & d + rb \end{vmatrix}, \\ &= a(d + rb) - b(c + ra), \\ &= ad - bc, \\ &= |A|.\end{aligned}$$

To prove part (2), craft B by swapping rows 1 and 2 of matrix A.

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

or,

$$Ay_p + \mathbf{f} = \begin{pmatrix} (-3a_1 + 6a_2)t + (-3b_1 + 6b_2 + 3) \\ (-2a_1 + 4a_2)t + (-2b_1 + 4b_2 + 4) \end{pmatrix}.$$

Comparing coefficients of the polynomial entries (e.g., $0 = -3a_1 + 6a_2$ and $a_1 = -3b_1 + 6b_2 + 3$), we get the following system.

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ a_1 + 3b_1 - 6b_2 &= 3 \\ a_2 + 2b_1 - 4b_2 &= 4 \end{aligned}$$

Solving, $a_1 = -12$, $a_2 = -6$, $b_1 = 5 + 2b_2$, with b_2 free. Letting $b_2 = 0$, we get $a_1 = -12$, $a_2 = -6$, $b_1 = 5$, and $b_2 = 0$, providing the particular solution

$$\mathbf{y}_p = \begin{pmatrix} -12 \\ -6 \end{pmatrix} t + \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

17. If $\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t}$, then

$$\begin{aligned} \mathbf{y}_p' &= \mathbf{a}e^{-t} - (\mathbf{a}t + \mathbf{b})e^{-t} = [-\mathbf{a}t + (\mathbf{a} - \mathbf{b})]e^{-t} \\ &= \begin{pmatrix} -a_1 t + (a_1 - b_1) \\ -a_2 t + (a_2 - b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Next,

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= A(\mathbf{a}t + \mathbf{b})e^{-t} + \mathbf{f} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \\ &= \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Thus,

$$\begin{pmatrix} -a_1 t + (a_1 - b_1) \\ -a_2 t + (a_2 - b_2) \end{pmatrix} = \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix}.$$

Comparing coefficients of the polynomial entries in these vectors (e.g., $-a_1 = (a_1 + 2a_2)$) leads to the system

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_1 - 2b_1 - 2b_2 &= 1 \\ a_2 - 2b_1 - 2b_2 &= 0. \end{aligned}$$

The solution of this system is $a_1 = 1/2$, $a_2 = -1/2$, $b_1 = -1/4 - b_2$, where b_2 is free. Choosing $b_2 = -1/8$ provides the solution $a_1 = 1/2$, $a_2 = -1/2$, $b_1 = -1/8$, and $b_2 = -1/8$, so

$$\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t} = \begin{pmatrix} (1/2)t - 1/8 \\ (-1/2)t - 1/8 \end{pmatrix} e^{-t},$$

which is identical to that found in Example 9.15.

18. The current coming into the node at "a" must equal the current coming out of the same node. Hence,

$$i = i_1 + i_2. \tag{9.16}$$

Traversing (clockwise) the leftmost loop containing emf, resistor, and inductor, Kirchoff's voltage law provides

$$-30 + 10i + 0.02i_1' = 0. \tag{9.17}$$

Traversing (clockwise) the outer loop containing emf, both resistors, and the far inductor,

$$-30 + 10i + 20i_2 + 0.04i_2' = 0. \tag{9.18}$$

If we now subsitute $i = i_1 + i_2$ into equations (9.17) and (9.18), then a little algebra provides us with the following system (in the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$).

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}' = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix} \tag{9.19}$$

A computer provides the following eigenvalue-eigenvector pairs for the coefficient matrix in equation (9.19):

$$-1000 \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad -250 \longrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence, the homogeneous solution is

$$\mathbf{x} = C_1 e^{-1000t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-250t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \tag{9.20}$$

Now for a particular solution, let's try the form $\mathbf{x}_p = (a_1, a_2)^T$. Substituting this informed guess in equation (9.19), we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix}.$$

Choose $y_1(x)$ with $1 = y_1(0) = a_0$ and $0 = y_1'(0) = a_1$.

$$y_1(x) = 1 + \frac{1}{2\cdot 1}x^2 - \frac{3}{4\cdot 3\cdot 2\cdot 1}x^4 + \cdots = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}(3\cdot 7\cdot 11\cdot\cdots\cdot(4n-5))}{(2n)!}x^{2n}$$

Choose $y_2(x)$ with $0 = y_2(0) = a_0$ and $1 = y_2'(0) = a_1$.

$$y_2(x) = x - \frac{1}{3\cdot 2}x^3 + \frac{5}{5\cdot 4\cdot 3\cdot 2}x^5 - \cdots = x - \frac{1}{6}x^3 + \sum_{n=2}^{\infty}\frac{(-1)^n(5\cdot 9\cdot 12\cdot\cdots\cdot(4n-3))}{(2n+1)!}x^{2n+1}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

18. The coefficients of y' and y are $p(x) = x$ and $q(x) = 2$, both polynomials and analytic at $x = 0$. Thus, $x = 0$ is an ordinary point. According to Theorem 2.29, all solutions have radius of convergence $R = \infty$. We seek a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with derivatives

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \quad\text{and}\quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 = y'' + xy' + 2y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned}$$

Shifting the index of the first term and noting that the second term

$$\sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n,$$

we write

$$0 = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+2)a_n\right]x^n.$$

Setting coefficients equal to zero,

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0 \quad\text{or}\quad a_{n+2} = \frac{-a_n}{n+1}, \quad n \geq 0.$$

Thus,

$$a_2 = -a_0, \quad a_4 = \frac{-a_2}{3} = \frac{a_0}{3}, \quad a_6 = \frac{-a_4}{5} = \frac{-a_0}{5\cdot 3}, \quad\text{and}\quad a_8 = \frac{-a_6}{7} = \frac{a_0}{7\cdot 5\cdot 3}.$$

Section 13.2. Separation of Variables for the Heat Equation

1. The thermal diffusivity of gold is $k = 1.18\,\text{cm}^2/\text{sec}$. We will let the unit of length be centimeters, so $L = 50$. The boundary conditions are $u(0, t) = 0$ and $u(50, t) = 0$, so the steady-state temperature is 0. Hence the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L),$$

where the coefficients are

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L 100 \sin(n\pi x/L)\,dx \\ &= \frac{200}{n\pi}[1 - \cos(n\pi)] \\ &= \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

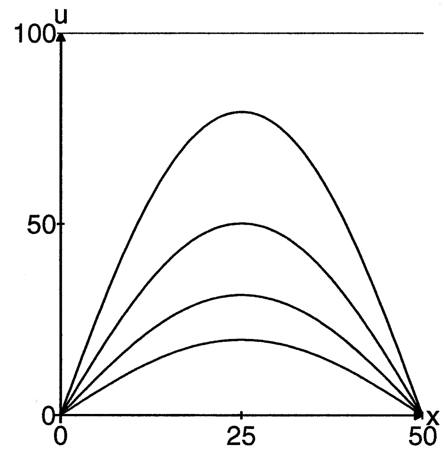
Thus the solution is given by

$$\begin{aligned} u(x, t) = \sum_{p=0}^{\infty} &\frac{400}{(2p+1)\pi} e^{-1.18\times(2p+1)^2\pi^2 t/2500} \\ &\times \sin\left(\frac{n\pi x}{50}\right). \end{aligned}$$

For $t = 100$, the term for $p = 1$ is bounded by 0.65. Hence one term of the series will suffice to estimate the temperature within 1°. Using just this one term, we solve

$$\frac{400}{\pi} e^{-1.18\times\pi^2 t/2500} = 10$$

for $t = 546$ sec. Hence we see that it will take about 546 sec for the temperature to drop below 10°C. The temperature at 100 second intervals is plotted below.



2. The thermal diffusivity of aluminum is $k = 0.84\,\text{cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 2.14, while that for $p = 2$ is bounded by 0.006. Hence two terms will suffice to compute the temperature for $t = 100$ sec. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-0.84\times\pi^2 t/2500} = 10$$

for $t = 767$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 767 sec for the temperature to drop below 10°C.

The thermal diffusivity of silver is $k = 1.7\,\text{cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 0.1.Hence one term will suffice to compute the temperature for $t = 100$ sec. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-1.7\times\pi^2 t/2500} = 10$$

for $t = 379$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 379 sec for the temperature to drop below 10°C.