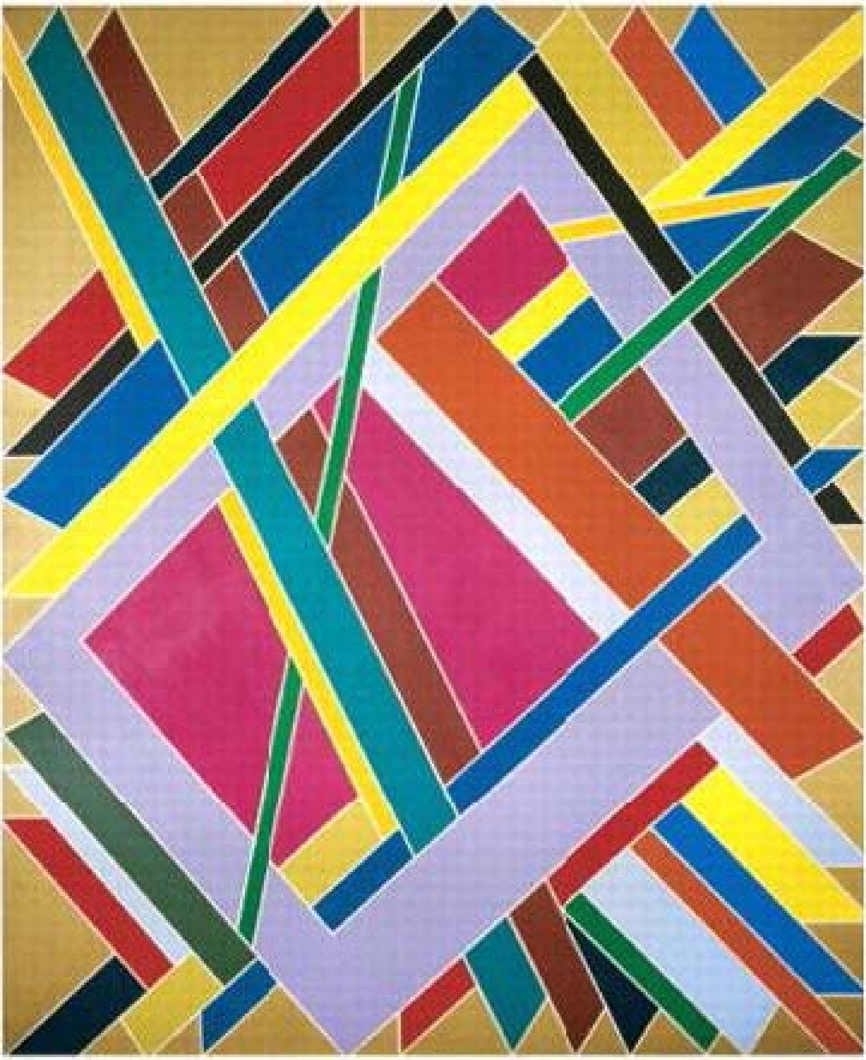
ELEMENTARY LINEAR ALGEBRA

WITH APPLICATIONS

NINTH EDITION



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It is not difficult to prove that performing these manipulations on a linear system leads to an equivalent system. The next example proves this for the second type of manipulation. Exercises 24 and 25 prove it for the first and third manipulations, respectively.

EXAMPLE 6

Suppose that the ith equation of the linear system (2) is multiplied by the nonzero constant c, producing the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ ca_{i1}x_1 + ca_{i2}x_2 + \cdots + ca_{in}x_n &= cb_i \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{17}$$

If $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution to (2), then it is a solution to all the equations in (17), except possibly to the ith equation. For the ith equation we have

$$c(a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n) = cb_i$$

or

$$ca_{i1}s_1 + ca_{i2}s_2 + \cdots + ca_{in}s_n = cb_i.$$

Thus the ith equation of (17) is also satisfied. Hence every solution to (2) is also a solution to (17). Conversely, every solution to (17) also satisfies (2). Hence (2) and (17) are equivalent systems. ■

The following example gives an application leading to a linear system of two equations in three unknowns:

EXAMPLE 7

(Production Planning) A manufacturer makes three different types of chemical products: A, B, and C. Each product must go through two processing machines: X and Y. The products require the following times in machines X and Y:

1. One ton of A requires 2 hours in machine X and 2 hours in machine Y.
2. One ton of B requires 3 hours in machine X and 2 hours in machine Y.
3. One ton of C requires 4 hours in machine X and 3 hours in machine Y.

Machine X is available 80 hours per week, and machine Y is available 60 hours per week. Since management does not want to keep the expensive machines X and Y idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of the products as is made.

To solve this problem, we let x_1, x_2, and x_3 denote the number of tons of products A, B, and C, respectively, to be made. The number of hours that machine X will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80. Thus we have

$$2x_1 + 3x_2 + 4x_3 = 80.$$

We shall sometimes use the **summation notation**, and we now review this useful and compact notation.

By $\sum_{i=1}^{n} a_i$ we mean $a_1 + a_2 + \cdots + a_n$. The letter i is called the **index of summation**; it is a dummy variable that can be replaced by another letter. Hence we can write

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

Thus

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4.$$

The summation notation satisfies the following properties:

1. $\sum_{i=1}^{n} (r_i + s_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$
2. $\sum_{i=1}^{n} c(r_i a_i) = c \sum_{i=1}^{n} r_i a_i$
3. $\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)$

Property 3 can be interpreted as follows: The left side is obtained by adding all the entries in each column and then adding all the resulting numbers. The right side is obtained by adding all the entries in each row and then adding all the resulting numbers.

If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k \tag{2}$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$, and $c_1, c_2, \ldots, c_k$ are called **coefficients**.

The linear combination in Equation (2) can also be expressed in summation notation as

$$\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots + c_k A_k.$$

EXAMPLE 13 The following are linear combinations of matrices:

$$3 \begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix},$$

$$2 \begin{bmatrix} 3 & -2 \end{bmatrix} - 3 \begin{bmatrix} 5 & 0 \end{bmatrix} + 4 \begin{bmatrix} -2 & 5 \end{bmatrix},$$

$$-0.5 \begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4 \begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}.$$

columns of matrix A, where the coefficients are the entries in the jth column of matrix B:

$$\operatorname{col}_j(AB) = A\operatorname{col}_j(B) = b_{1j}\operatorname{col}_1(A) + b_{2j}\operatorname{col}_2(A) + \cdots + b_{pj}\operatorname{col}_p(A).$$

EXAMPLE 13

If A and B are the matrices defined in Example 11, then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 6 \\ 6 & 17 & 16 \\ 17 & 7 & 1 \end{bmatrix}.$$

The columns of AB as linear combinations of the columns of A are given by

$$\operatorname{col}_1(AB) = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix} = A\operatorname{col}_1(B) = -2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\operatorname{col}_2(AB) = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix} = A\operatorname{col}_2(B) = 3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\operatorname{col}_3(AB) = \begin{bmatrix} 6 \\ 16 \\ 1 \end{bmatrix} = A\operatorname{col}_3(B) = 4\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

■

■ Linear Systems

Consider the linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{4}$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

and (verify)

$$AC + BC = \begin{bmatrix} 15 & 1 \\ 7 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 12 & 3 \end{bmatrix}.$$

■

Recall Example 9 in Section 1.3, which shows that AB need not always equal BA. This is the first significant difference between multiplication of matrices and multiplication of real numbers.

Theorem 1.3 **Properties of Scalar Multiplication**

If r and s are real numbers and A and B are matrices of the appropriate sizes, then

(a) $r(sA) = (rs)A$

(b) $(r + s)A = rA + sA$

(c) $r(A + B) = rA + rB$

(d) $A(rB) = r(AB) = (rA)B$

Proof

Exercises 13, 14, 16, and 18. ■

EXAMPLE 5 Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Then

$$2(3A) = 2\begin{bmatrix} 12 & 6 & 9 \\ 6 & -9 & 12 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 18 \\ 12 & -18 & 24 \end{bmatrix} = 6A.$$

We also have

$$A(2B) = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 6 & -4 & 2 \\ 4 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 32 & -10 & 16 \\ 0 & 0 & 26 \end{bmatrix} = 2(AB).$$

■

EXAMPLE 6 Scalar multiplication can be used to change the size of entries in a matrix to meet prescribed properties. Let

$$A = \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix}.$$

Then for $k = \frac{1}{7}$, the largest entry of kA is 1. Also if the entries of A represent the volume of products in gallons, for $k = 4$, kA gives the volume in quarts. ■

So far we have seen that multiplication and addition of matrices have much in common with multiplication and addition of real numbers. We now look at some properties of the transpose.

Supplementary Exercises

1. Let

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

Find a matrix B in reduced row echelon form that is row equivalent to A, using elementary matrices.

2. Find all values of a for which the following linear systems have solutions:

(a) $$\begin{aligned} x + 2y + z &= a^2 \\ x + y + 3z &= a \\ 3x + 4y + 7z &= 8 \end{aligned}$$

(b) $$\begin{aligned} x + 2y + z &= a^2 \\ x + y + 3z &= a \\ 3x + 4y + 8z &= 8 \end{aligned}$$

3. Find all values of a for which the following homogeneous system has nontrivial solutions:

$$\begin{aligned} (1-a)x \qquad + z &= 0 \\ -ay + z &= 0 \\ y + z &= 0 \end{aligned}$$

4. Find all values of a, b, and c so that the linear system

$$A\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is consistent for

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -2 \\ 1 & 3 & -10 \end{bmatrix}.$$

5. Let A be an $n \times n$ matrix.

(a) Suppose that the matrix B is obtained from A by multiplying the jth row of A by $k \neq 0$. Find an elementary row operation that, when applied to B, gives A.

(b) Suppose that the matrix C is obtained from A by interchanging the ith and jth rows of A. Find an elementary row operation that, when applied to C, gives A.

(c) Suppose that the matrix D is obtained from A by adding k times the jth row of A to its ith row. Find an elementary row operation that, when applied to D, gives A.

6. Exercise 5 implies that the effect of any elementary row operation can be reversed by another (suitable) elementary row operation.

(a) Suppose that the matrix E_1 is obtained from I_n by multiplying the jth row of I_n by $k \neq 0$. Explain why E_1 is nonsingular.

(b) Suppose that the matrix E_2 is obtained from I_n by interchanging the ith and jth rows of I_n. Explain why E_2 is nonsingular.

(c) Suppose that the matrix E_3 is obtained from I_n by adding k times the jth row of I_n to its ith row. Explain why E_3 is nonsingular.

7. Find the inverse of

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

8. Find the inverse of

$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

9. As part of a project, two students must determine the inverse of a given 10×10 matrix A. Each performs the required calculation, and they return their results A_1 and A_2, respectively, to the instructor.

(a) What must be true about the two results? Why?

(b) How does the instructor check their work without repeating the calculations?

10. Compute the vector $\mathbf{w}$ for each of the following expressions without computing the inverse of any matrix, given that

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 2 \\ -1 & 1 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ 7 \\ -3 \end{bmatrix}:$$

(a) $\mathbf{w} = A^{-1}(C + F)\mathbf{v}$ **(b)** $\mathbf{w} = (F + 2A)C^{-1}\mathbf{v}$

11. Determine all values of s so that

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & s & 2 \end{bmatrix}$$

is nonsingular.

12. Determine all values of s so that

$$A = \begin{bmatrix} s & 1 & 0 \\ 1 & s & 1 \\ 0 & 1 & s \end{bmatrix}$$

is nonsingular.

Key Terms

Vectors	Tail of a vector	Scalar multiple of a vector
Rectangular (Cartesian) coordinate system	Head of a vector	Vector addition
Coordinate axes	Directed line segment	Zero vector
x-axis, y-axis, z-axis	Magnitude of a vector	Difference of vectors
Origin	Vector in the plane	Right- (left-) handed coordinate system
Coordinates	Components of a vector	3-space, R^3
2-space, R^2	Equal vectors	Vector in space

4.1 Exercises

1. Sketch a directed line segment in R^2, representing each of the following vectors:

(a) $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **(b)** $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(c) $\mathbf{w} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ **(d)** $\mathbf{z} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

2. Determine the head of the vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ whose tail is $(-3, 2)$. Make a sketch.

3. Determine the tail of the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ whose head is $(1, 2)$. Make a sketch.

4. Determine the tail of the vector $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ whose head is $(3, -2, 2)$.

5. For what values of a and b are the vectors $\begin{bmatrix} a-b \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ a+b \end{bmatrix}$ equal?

6. For what values of a, b, and c are the vectors $\begin{bmatrix} 2a-b \\ a-2b \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ a+b-2c \end{bmatrix}$ equal?

In Exercises 7 and 8, determine the components of each vector $\overrightarrow{PQ}$.

7. **(a)** $P(1, 2)$, $Q(3, 5)$

(b) $P(-2, 2, 3)$, $Q(-3, 5, 2)$

8. **(a)** $P(-1, 0)$, $Q(-3, -4)$

(b) $P(1, 1, 2)$, $Q(1, -2, -4)$

In Exercises 9 and 10, find a vector whose tail is the origin that represents each vector $\overrightarrow{PQ}$.

9. **(a)** $P(-1, 2)$, $Q(3, 5)$

(b) $P(1, 1, -2)$, $Q(3, 4, 5)$

10. **(a)** $P(2, -3)$, $Q(-2, 4)$

(b) $P(-2, -3, 4)$, $Q(0, 0, 1)$

11. Compute $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, and $3\mathbf{u} - 2\mathbf{v}$ if

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$;

(b) $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

(c) $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

12. Compute $\mathbf{u} + \mathbf{v}$, $2\mathbf{u} - \mathbf{v}$, $3\mathbf{u} - 2\mathbf{v}$, and $\mathbf{0} - 3\mathbf{v}$ if

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$;

(b) $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$;

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

13. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$c = -2$, and $d = 3$. Compute each of the following:

(a) $\mathbf{u} + \mathbf{v}$

(b) $c\mathbf{u} + d\mathbf{w}$

(c) $\mathbf{u} + \mathbf{v} + \mathbf{w}$

(d) $c\mathbf{u} + d\mathbf{v} + \mathbf{w}$

3. Which of the following can be transition matrices of a Markov process?

(a) $\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.8 & 0.5 & 0.7 \\ 0.0 & 0.2 & 0.2 \end{bmatrix}$

(c) $\begin{bmatrix} 0.55 & 0.33 \\ 0.45 & 0.67 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.8 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$

4. Which of the following are probability vectors?

(a) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$ **(d)** $\begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{1}{10} \\ \frac{2}{10} \end{bmatrix}$

5. Consider the transition matrix

$$T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

(a) If $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ to three decimal places.

(b) Show that T is regular and find its steady-state vector.

6. Consider the transition matrix

$$T = \begin{bmatrix} 0 & 0.2 & 0.0 \\ 0 & 0.3 & 0.3 \\ 1 & 0.5 & 0.7 \end{bmatrix}.$$

(a) If

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$, and $\mathbf{x}^{(4)}$ to three decimal places.

(b) Show that T is regular and find its steady-state vector.

7. Which of the following transition matrices are regular?

(a) $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ **(b)** $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} \frac{1}{4} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$

8. Show that each of the following transition matrices reaches a state of equilibrium:

(a) $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{4} \end{bmatrix}$ **(d)** $\begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$

9. Find the steady-state vector of each of the following regular matrices:

(a) $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix}$ **(b)** $\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ \frac{3}{4} & 0 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0.4 & 0.0 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.6 \end{bmatrix}$

10. (***Psychology***) A behavioral psychologist places a rat each day in a cage with two doors, A and B. The rat can go through door A, where it receives an electric shock, or through door B, where it receives some food. A record is made of the door through which the rat passes. At the start of the experiment, on a Monday, the rat is equally likely to go through door A as through door B. After going through door A and receiving a shock, the probability of going through the same door on the next day is 0.3. After going through door B and receiving food, the probability of going through the same door on the next day is 0.6.

(a) Write the transition matrix for the Markov process.

(b) What is the probability of the rat going through door A on Thursday (the third day after starting the experiment)?

(c) What is the steady-state vector?

11. (***Sociology***) A study has determined that the occupation of a boy, as an adult, depends upon the occupation of his father and is given by the following transition matrix where P = professional, F = farmer, and L = laborer:

$$\text{Son's occupation} \quad \begin{array}{c} \\ \\ \text{P} \\ \text{F} \\ \text{L} \end{array} \begin{array}{c} \text{Father's occupation} \\ \begin{array}{ccc} \text{P} & \text{F} & \text{L} \end{array} \\ \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

N