INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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27 (a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's !

28 One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

29 The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{y}\\-\boldsymbol{y}\end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

30 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

31 (a) rank 1 (b) rank 2 because every row is a combination of $(1,2,3,4)$ and $(1,1,1,1)$

(c) rank 1 because all columns are multiples of $(1,1,1)$

32 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1,0,0,1,-1,0)$ and $(0,0,-1,-1,0,1)$ and $(0,-1,0,0,1,-1)$. Those are flows around the 3 small loops.

33 (a) and (c) are correct; (b) is completely false; (d) is false because R might have 1's in nonpivot columns.

34 $R_A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ $R_B = \begin{bmatrix}R_A & R_A\end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix}R_A & 0\\0 & R_A\end{bmatrix} \longrightarrow$ Zero rows go to the bottom

35 If all pivot variables come last then $R = \begin{bmatrix}0 & I\\0 & 0\end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix}I\\0\end{bmatrix}$.

36 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

37 A and A^{T} have the same rank $r =$ number of pivots. But *pivcol* (the column number) is 2 for this matrix A and 1 for A^{T}: $A = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$.

38 Special solutions in $N = [\,-2 \quad -4 \quad 1 \quad 0;\ -3 \quad -5 \quad 0 \quad 1\,]$ and $[\,1 \quad 0 \quad 0;\ 0 \quad -2 \quad 1\,]$.

$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{6} & \mathbf{-11} & \mathbf{6} \end{bmatrix}$ Notice the trace $6 = 1+2+3$ and determinant $6 = (1)(2)(3)$.

21 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. Pick $M = A - \lambda I$.

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have *different eigenvectors*.

22 The eigenvalues must be $\lambda = \mathbf{1}$ (because the matrix is Markov), $\mathbf{0}$ (for singular), $-\frac{\mathbf{1}}{\mathbf{2}}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.30.

24 $\lambda = \mathbf{0, 0, 6}$ (*notice rank* 1 *and trace* 6). Two eigenvectors of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are perpendicular to $\boldsymbol{v}$ and the third eigenvector is $\boldsymbol{u}$: $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

25 When A and B have the same n λ's and $\boldsymbol{x}$'s, look at any combination $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. Multiply by A and B: $A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **equals** $B\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **for all vectors** $\boldsymbol{v}$. So $A = B$.

26 The block matrix has $\lambda = 1$, 2 from B and $\lambda = 5$, 7 from D. All entries of C are multiplied by zeros in $\det(A - \lambda I)$, so C has no effect on the eigenvalues of the block matrix.

27 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

28 Subtract from $0, 0, 0, 4$ in Problem 27. $B = A - I$ has $\lambda = -1, -1, -1, 3$ and $C = I - A$ has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

29 A is triangular : $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; C has rank one : $\lambda(C) = 0, 0, 6$.

eig(K, M) without actually inverting M.) All eigenvalues λ are positive:

$$ST\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (T\boldsymbol{x})^{\mathrm{T}}ST\boldsymbol{x} = (T\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}T^{\mathrm{T}}ST\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0.$$

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2-\sqrt{3},\ 2-1,\ 2,\ 2+1,\ 2+\sqrt{3}$. The product of those eigenvalues is $6 = \det K$.

35 Put parentheses in $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \mathbf{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \mathbf{0}$ only happens when $\boldsymbol{x} = \mathbf{0}$. Thus $A^{\mathrm{T}}CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}}CA$ in a wide range of applications. I believe this is a unifying concept from linear algebra.

36 (a) The eigenvectors of $\lambda_1 I - S$ are $\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$. Those are ≥ 0; $\lambda_1 I - S$ is semidefinite.

(b) Semidefinite matrices have energy $\boldsymbol{x}^{\mathrm{T}}(\lambda_1 I - S)\,\boldsymbol{x}_2 \geq 0$. Then $\lambda_1\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$.

(c) Part (b) says $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq \lambda_1$ for all $\boldsymbol{x}$. Equality at the eigenvector with $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$.

37 Energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = a\,(x_1+x_2+x_3)^2 + c\,(x_2-x_3)^2 \geq 0$ if $a \geq 0$ and $c \geq 0$: semidefinite. The matrix has rank ≤ 2 and determinant $= 0$; cannot be positive definite for any a and c.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

2 $A = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$ is similar to $B = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

Problem Set 10.3, page 480

1 Eigenvalues $\lambda = 1$ and $.75$; $(A - I)\boldsymbol{x} = 0$ gives the steady state $\boldsymbol{x} = (.6, .4)$ with $A\boldsymbol{x} = \boldsymbol{x}$.

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4 A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$, because each row of A^{T} adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A.)

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.

6 Add the components of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

8 If $P =$ cyclic permutation and $\boldsymbol{u}_0 = (1, 0, 0, 0)$ then $\boldsymbol{u}_1 = (0, 0, 1, 0)$; $\boldsymbol{u}_2 = (0, 1, 0, 0)$; $\boldsymbol{u}_3 = (1, 0, 0, 0)$; $\boldsymbol{u}_4 = \boldsymbol{u}_0$. The eigenvalues $1, i, -1, -i$ are all *on the unit circle*. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue $\lambda = 1$.

9 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply on the right by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

11 Last row $.2, .3, .5$ makes $A = A^{\mathrm{T}}$; rows also add to 1 so $(1, \ldots, 1)$ is also an eigenvector of A.

12 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, \ .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

13 $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\boldsymbol{x} = (.9, .9, .9)$