

CHAPTER I

Chapter I PROBLEMS

1. (a) $\Omega = \{(B,W), (B,G), (G,W), (G,G)\}$. The sample space contains four outcomes; an outcome itself is a 2-tuple where the first component represents the result of drawing from urn one and the second component from urn two.

- (b) The event space is the collection all subsets of the sample space. There are 16 such subsets.

$$\mathcal{A} = \{\emptyset, \Omega, \{(B,W)\}, \{(B,G)\}, \{(G,W)\}, \{(G,G)\}, \{(B,W), (B,G)\}, \{(B,W), (G,W)\}, \{(B,W), (G,G)\}, \{(B,G), (G,W)\}, \{(B,G), (G,G)\}, \{(G,W), (G,G)\}, \{(B,W), (B,G), (G,W)\}, \{(B,W), (B,G), (G,G)\}, \{(B,W), (G,W), (G,G)\}, \{(B,G), (G,W), (G,G)\}\}$$

- (c) $1/4$

- (d) 0

2. (a) There are many ways to describe the outcomes of this experiment. For example, one could number the balls in urn one as 1, 2, 3 red; 4, 5 white; and 6 blue and those in urn two as 1 red, 2, 3 white; and 4, 5, 6 blue.

- (i) Then $\Omega = \{(i_1, i_2) : i_1 = 1, \dots, 6 \text{ and } i_2 = 1, \dots, 6, \text{ where } i_1 \text{ is the number on the ball drawn from urn 1 and } i_2 \text{ is the number on the ball drawn from urn 2.}\}$

Note that there are 36 outcomes of this experiment.

- (ii) Let A denote the event both balls are red, B denote the event both balls are white, and C denote the event both balls are blue.

$$\text{Then } P[\text{both balls same color}] = P[A \cup B \cup C] = P[A] + P[B] + P[C] = \frac{3}{36} + \frac{4}{36} + \frac{3}{36}.$$

$$(iii) P[A] = \frac{3}{36} < \frac{4}{36} = P[B].$$

$$(b) (i) \frac{12 \cdot 8 \cdot 4}{12^3} \quad (ii) \frac{12 \cdot 8 \cdot 4}{12 \cdot 11 \cdot 10}$$

(2) Show that $\sum_{j=0}^n \binom{n}{j} (p_1 q_1^{n-j} - p_2 q_2^{n-j}) = \sum_{j=0}^n d_j$ (say) ≥ 0 .

Note that $\sum_{j=0}^n d_j = 0$, hence it suffices to show that the first few d_j 's

are positive, and the remaining are negative. But $d_j \geq 0$ if and only if

$$j \leq \log(q_2/q_1) / \log(p_1 q_2 / p_2 q_1).$$

(Use the result of Problem 28 for an alternate proof.)

8. $\sum_{j=60}^{100} \frac{\binom{2500}{j} \binom{2500}{100-j}}{\binom{5000}{100}}$. The hypergeometric can be approximated by the binomial

and the binomial can in turn be approximated by the normal which gives a numerical answer of approximately $1 - \Phi(2) = .0228$

11. Let X denote the number of defectives in the sample. Assume that X has a binomial distribution.

(a) $P[X \geq 1] = 1 - P[X = 0] = 1 - (.99)^{10}$.

(b) Want $P[X \geq 1] \approx .95$; or, want $P[X = 0] \approx .05$;

i.e., $(.9)^n \approx .05$, or, $n \approx 29$.

15. $p + \phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma}) / [\phi(\frac{b-\mu}{\sigma}) - \phi(\frac{a-\mu}{\sigma})]$

17. There is a misprint in this problem. The mean was intended to be 200 rather than

20. Want

$P[X \leq 150] \geq .90$, i.e., $\Phi(\frac{50}{\sigma}) \geq .90$, which implies $\sigma \approx 50/1.282 \approx 39$.

19. (a) $E[X] = \int_0^\infty x^{-2} \exp[-(1/2)(x/\beta)^2] dx$
 $= (1/2) \sqrt{2\pi} \beta^{-1} \int_0^\infty x (1/\beta \sqrt{2\pi}) \exp[-(1/2)(x/\beta)^2] dx$
 $= \beta \sqrt{2\pi}/2$ by recognizing that the last integral is the variance of a

normal distribution with mean 0 and variance β^2 , which shows how a little knowledge of probability can be an aid to integration.

$\text{var}[X] = \beta^2(4-\pi)/2$.

(b) No.

25.

1	2	3	4	5	6	7	8	9
$\frac{9}{81}$	$\frac{12}{81}$	$\frac{16}{81}$	$\frac{12}{81}$	$\frac{10}{81}$	$\frac{6}{81}$	$\frac{3}{81}$	$\frac{1}{81}$	

28. Assume true and differentiate both sides with respect to p to obtain the equality:

$$\sum_{j=k}^n j \binom{n}{j} p^{j-1} q^{n-j} - \sum_{j=k}^n (n-j) \binom{n}{j} p^j q^{n-j-1} = k \binom{n}{k} p^{k-1} q^{n-k}.$$

The inequality is verified by noting the $(j+1)$ st term of the first sum cancels the j th term of the second sum. Work backwards.

29. Let $X = \#$ of successes in first n Bernoulli trials

and $Y = \#$ of failures prior to r th success.

Note that $\{X \leq r-1\} \equiv \{Y > n-r\}$ hence $\Gamma_X(r-1) = P[X \leq r-1] = 1 - P[Y > n-r] = 1 - \Gamma_Y(n-r)$.

30. $E[Z_\lambda] = \{E[U^\lambda] - E[(1-U)^\lambda]\} / \lambda = 0$ for $\lambda > -1$.

$E[Z_\lambda^2] = \{E[U^{2\lambda}] - 2E[U^\lambda(1-U)^\lambda] + E[(1-U)^{2\lambda}]\} / \lambda^2$

$= (2/\lambda^2) \{[1/(2\lambda+1)] - B(\lambda+1, \lambda+1)\}$ for $\lambda > -1/2$.

$E[Z_\lambda^3] = 0$ for $\lambda > -1/3$.

$E[Z_\lambda^4] = (2/\lambda^4) \{[1/(4\lambda+1)] - 4B(3\lambda+1, \lambda+1) + 3B(2\lambda+1, 2\lambda+1)\}$ for $\lambda > -1/4$.

The last part is misstated. The intent was to get two different λ 's,

say λ_1 and λ_2 , such that Z_{λ_1} and Z_{λ_2} have the same skewness and kurtosis.

If λ_1 and λ_2 are sought so that Z_{λ_1} and Z_{λ_2} have kurtosis equal to zero,

then $\lambda_1 \approx .135$ and $\lambda_2 \approx 5.20$ will work.

$$30. (a) P(X=x, Y=y) = \frac{\binom{4}{x} \binom{4}{y} \binom{4}{6-x-y}}{\binom{52}{6}}$$

$$30. (a) (26 - 9x)/(9 - 3x)$$

$$(a) E[XY|X=x] = xE[Y|X=x].$$

30. No

$$32. m_{Y|X=x}(t) = E[e^{tY}|X=x]. \quad m_Y(t) = E[e^{tY}] = E[E[e^{tY}|X]] = E[m_{Y|X}(t)].$$

$$33. (b) 1 \quad (c) \rho_{X,Y} = 1/2 \quad (d) f_X(x)f_Y(y)$$

$$34. (a) f(Y) = E[f(Y|X)] = E[X+1/2] = 1$$

$$(b) \text{cov}(X,Y) = 1/12.$$

$$(c) 1/4.$$

35. Special case of Problem 46.

36. The joint density of X and Y might have two, three, or four mass points. Consider the case of four mass points. Let $p_{ij} = P(X=x_i; Y=y_j)$ for $i, j = 1, 2$, where $x_1 < x_2$ and $y_1 < y_2$.

$$\text{Write } p_{1.} = p_{11} + p_{12} = P(X=x_1),$$

$$p_{2.} = p_{21} + p_{22} = P(X=x_2),$$

$$p_{.1} = p_{11} + p_{21} = P(Y=y_1), \text{ and}$$

$$p_{.2} = p_{12} + p_{22} = P(Y=y_2).$$

$$\text{Let } U = (X-x_1)/(x_2-x_1) \text{ and } V = (Y-y_1)/(y_2-y_1).$$

Now $\text{cov}(X,Y) = 0$ if and only if $\text{cov}(U,V) = 0$ and X and Y are independent if and only if U and V are independent.

$$\text{cov}(U,V) = E[UV] - E[U]E[V] = p_{22} - p_{2.}p_{.2}.$$

$\text{cov}(U,V) = 0$ implies $p_{22} = p_{2.}p_{.2}$ which in turn implies independence.

Chapter V PROBLEMS

$$1. (a) \text{cov}(X_1 + X_2, X_2 + X_3) = \sigma^2; \text{var}[X_1 + X_2] = \text{var}[X_2 + X_3] = 2\sigma^2; \\ \text{hence } \rho[X_1 + X_2, X_2 + X_3] = 1/2.$$

$$(b) (\sigma_2^2 - \sigma_1^2)/(\sigma_1^2 + \sigma_2^2).$$

$$(c) 1/2.$$

$$3. F(x)I_{[0,\infty)}(x).$$

$$4. (a) P(X=x) = \frac{\binom{M-K}{x-1}}{\binom{M}{x-1}} \cdot \frac{K}{M-x+1} \quad \text{for } x = 1, \dots, M-K+1.$$

$$(b) P(Z=z) = \frac{\binom{K}{r-1} \binom{M-K}{z-r}}{\binom{M}{z-1}} \cdot \frac{\binom{K-r+1}{1}}{\binom{M-z+1}{1}} \quad \text{for } z = r, \dots, M-K+r.$$

$$(c) \frac{(x,y)}{f_{X,Y}(x,y)} \left| \begin{array}{c|c|c|c|c|c} (1,2) & (1,3) & (2,1) & (3,1) & (4,1) \\ \hline \frac{2}{5} \cdot \frac{3}{4} & \frac{2}{5} \cdot \frac{1}{4} & \frac{3}{5} \cdot \frac{2}{4} & \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} & \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \end{array} \right|$$

5. According to the definition of expectation, $E[X_1]$ does not exist; however, there is no harm in saying $E[X_1] = \infty$. $E[Y_1] = n/(n-1)$ for $n > 1$.

6. (a) Since $X \leq \max(X,Y)$, $E[X] \leq E[\max(X,Y)]$; similarly, $E[Y] \leq E[\max(X,Y)]$, hence $\max\{E[X], E[Y]\} \leq E[\max(X,Y)]$.

$$(b) \max(X,Y) + \min(X,Y) = X + Y.$$

7. (a) Note that X and Y are independent and uniformly distributed. Apply the corollary of Theorem 3 on page 180.

(b) Theorem 8 will do it.

8. The cdf of $Z = \max(X,Y)$ is given by

$$(1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z})I_{(0,\infty)}(z)$$

$$\text{so } E[Z] = E[\max(X,Y)] = \int_0^1 (1 - F_Z(z)) dz = \int_0^1 (e^{-\lambda_1 z} + e^{-\lambda_2 z} - e^{-(\lambda_1 + \lambda_2)z}) dz = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

9. $X_2 - X_1 \sim N(0,2)$. The distribution of $(X_2 - X_1)^2$ can be found using Example 19. Similarly for $Y_2 - Y_1$ and $(Y_2 - Y_1)^2$. They are independent so use Equation (26) to find the distribution of $Z^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$.

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CHAPTER VI

23. Don't forget that $Z_1 + Z_2$ and $Z_2 - Z_1$ are independent! Similarly for $X_1 + X_2$ and $X_2 - X_1$.
- (b) t -distribution with 2 degrees of freedom.
- (c) Chi-square with 3 degrees of freedom.
- (d) F distribution with 1 and 1 degrees of freedom.
25. Note that X_1 and X_2 are independent and identically distributed chi-square random variables with 2 degrees of freedom, so X_1/X_2 has an F distribution with 2 and 2 degrees of freedom.
27. $U \sim N(\mu, 1/E(1/\sigma_j^2))$
 $V = I(X_1 - U)^2/\sigma_1^2 = I(X_1 - \mu)^2/\sigma_1^2 - (U - \mu)^2 I(1/\sigma_j^2)$ which is a difference of two independent chi-square distributed r.v.'s, the first with n degrees of freedom, the second with 1 degree of freedom. The result follows using the moment generating function technique. What result does this reduce to if all σ_j^2 are equal?
29. The joint distribution of (\bar{X}, S_1^2, S_2^2) is easily obtained since they are independent. Make a transformation and integrate out the unwanted variable.
30. One could use Theorem 13. On the other hand, note that $Y_2 - Y_1 = |X_1 - X_2|$ and the distribution of $X_1 - X_2$ is known and it is easy to find the distribution of the absolute value of a random variable.
31. (a) $1 - P[\text{both less than median}] = 3/4$.
 (b) $1 - P[\text{all are less than median}] = 1 - (1/2)^n$.
32. $E[F(Y_1)]$ is wanted. $F(Y_1)$ has the same distribution as the smallest observation of a random sample of size n from a uniform distribution over the interval $(0,1)$.
33. $E[Y_1] = \mu - [(n-1)/(n+1)]/\sqrt{3} \sigma$
 $E[Y_n] = \mu + [(n-1)/(n+1)]/\sqrt{3} \sigma$
 $\text{var}[Y_1] = \text{var}[Y_n] = 12\sigma^2 n/[(n+1)^2(n+2)]$.
 $\text{cov}[Y_1, Y_n] = 12\sigma^2/[(n+1)^2(n+2)]$.

- (a) $E[Y_n - Y_1] = [(n-1)/(n+1)]2\sqrt{3} \sigma$.
 $\text{var}[Y_n - Y_1] = 24\sigma^2(n-1)/[(n+1)^2(n+2)]$.
- (b) $E[(Y_1 + Y_n)/2] = \mu$.
 $\text{var}[(Y_1 + Y_n)/2] = 6\sigma^2/[(n+1)(n+2)]$
- (c) $E[Y_{k+1}] = \mu$.
 $\text{var}[Y_{k+1}] = 3\sigma^2/(2k+3)$.
- (d) $\frac{3\sigma^2}{n+2} > \frac{\sigma^2}{n} > \frac{6\sigma^2}{(n+1)(n+2)}$ for $n > 2$.

34. \bar{X} is asymptotically normally distributed with mean μ and variance $2\sigma^2/n$. The sample median is asymptotically normally distributed with mean μ and variance σ^2/n by Theorem 14. Note that the sample median has the smaller asymptotic variance.
35. $P[(Y_n - a_n)/b_n \leq y] = P[Y_n \leq b_n y + a_n] = \{1 - \exp[(b_n y - a_n)/(1 - b_n y - a_n)]\}^n = \{1 - \exp[\frac{y + (\log n)^2}{y - \log n}]\}^n$. Now let $n \rightarrow \infty$ and $\exp(-e^{-y})$ results.
36. (a) Similar to Problem 34.
 (b) With θ replacing λ choose a_n and b_n as in Example 9.
 (c) We know that $Y_1^{(n)}$ has exact distribution that is exponential with parameter $n\lambda$. So choose $a_n = 0$ and $b_n = 1/n$ and then $(Y_1^{(n)} - a_n)/b_n$ has exact (and hence also limiting) distribution that is exponential with parameter λ .

CHAPTER VIII

9. $(-2.09, 2.84)$ for σ known and $(-1.94, 2.69)$ for σ unknown.
10. (b) Use $\bar{X} - 1.645S$.
11. Use $Q = \sum_{i=1}^5 \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 / \sigma^2$ as your pivotal quantity. $Q \sim \text{chi-square}$ with 23 degrees of freedom.
12. Use $\frac{\sum_{i=1}^n (X_i - \bar{X})^2 / \sigma^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2 / \sigma^2} \sim F(m-1, n-1)$ as a pivotal quantity.
13. Want $P[2tS/\sqrt{20} < \sigma]$ where t is the $(1-\gamma)/2$ the quantile of a t -distribution with 19 degrees of freedom. Write $P[2tS/\sqrt{20} < \sigma] = P[(19)S^2/\sigma^2 < 19(20)/4t^2]$, where $(19)S^2/\sigma^2$ is chi-square distributed with 19 degrees of freedom, to complete the calculations for any γ .
14. (a) $2z\sigma/\sqrt{n}$ where z is the $(1+\gamma)/2$ quantile of a standard normal.
(b) $2t_\gamma[S]/\sqrt{n}$ where t is the $(1+\gamma)/2$ quantile of a t -distribution with $n-1$ degrees of freedom. See Problem 17 of Chapter VI for $\mathcal{L}[S]$.
15. Want $P[2tS/\sqrt{n} < \sigma/5] \approx .95$ where t is .95th quantile of a t -distribution with $n-1$ degrees of freedom. Rewrite as $P[(n-1)S^2/\sigma^2 < (n-1)n/100t^2]$.
Want the minimum n such that $(n-1)n \geq 100t_{.95, n-1}^2$. n a little over 300 seems to work.
16. Use Equation (10). $(1.47, 10.03)$
17. The first "the" should be "a". Use $Q = -I \log F(X_1; \theta) = -(1/\theta)I \log X_1$ as a pivotal quantity.
18. Use the statistical method and EX_1 as a statistic.
19. $[(Y_1 + Y_2)/2] - \theta$ is a good pivotal quantity.
20. The sample size seems large enough to use Equation (18) of Example 8.
.4375 \pm .0408 for 90%.
21. The UMVUE of $\tau(\theta)$ is a linear function of \bar{X} and S . \bar{X} and S are independent and have large sample normal distributions. Hence the large sample distribution of the UMVUE (or MLE) of $\tau(\theta)$ is normally distributed. Use this to get an approximate confidence interval.

26. Similar to Example 9.
27. The posterior distribution is given in the solution of Problem 45 of Chapter VII. Use it and Equation 21.
28. The likelihood function is the joint distribution of Y_1, \dots, Y_k looked at as a function of θ . $L(\theta; y_1, \dots, y_k) = \frac{n!}{(n-k)!} \theta^k e^{-\theta y_1} e^{-\theta y_k (n-k)}$ for y_1, y_2, \dots, y_k .
MLE of $1/\theta$ is $[\sum_{i=1}^k Y_i + (n-k)Y_k]/k$. Let $U_1 = Y_1 - Y_{1-1}$. $U_1 \sim \text{negative exponential}$ with parameter $\theta(n-1+1)$ using the lack of memory property of exponentially distributed random variables. $\theta(n-i+1)U_i \sim \text{negative exponential with parameter 1}$.
 $UY_1 + (n-k)Y_k = Y_1 + Y_2 + Y_3 + \dots + Y_{k-1} + (n-k+1)Y_k =$
 $U_1 + (U_1 + U_2) + (U_1 + U_2 + U_3) + \dots + (n-k+1)(U_1 + \dots + U_k) =$
 $nU_1 + (n-1)U_2 + \dots + (n-k+1)U_k = \sum_{j=1}^k (n-j+1)U_j$. Also,
 $\theta(UY_1 + (n-k)Y_k) = \sum_{j=1}^k \theta(n-j+1)U_j$, which is a sum of k independent negative exponentially distributed r.v.'s with parameter 1. Use $Q = \theta(UY_1 + (n-k)Y_k) \sim \text{gamma}(k, 1)$ as a pivotal quantity.