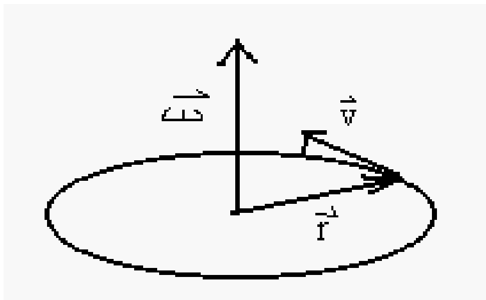
Chapter 1: Describing the universe

1. Circular motion. A particle is moving around a circle with angular velocity $\vec{\omega}$. Write its velocity vector $\vec{\mathbf{v}}$ as a vector product of $\vec{\omega}$ and the position vector $\mathbf{r}$ with respect to the center of the circle. Justify your expression. Differentiate your relation, and hence derive the angular form of Newton's second law ($\vec{\tau} = I\vec{\alpha}$) from the standard form (equation 1.8).



The direction of the velocity is perpendicular to $\vec{\omega}$ and also to the radius vector $\mathbf{r}$, and is given by putting your right thumb along the vector $\vec{\omega}$: your fingers then curl in the direction of the velocity. The speed is $v = \omega r$. Thus the vector relation we want is:

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

Differentiating, we get:

$$\begin{aligned}
\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} &= \frac{d\vec{\omega}}{dt} \times \vec{\mathbf{r}} + \vec{\omega} \times \frac{d\vec{\mathbf{r}}}{dt} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times \vec{\mathbf{v}} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}}) \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega}(\vec{\omega} \cdot \vec{\mathbf{r}}) - \omega^2 \vec{\mathbf{r}} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}
\end{aligned}$$

since $\vec{\omega}$ is perpendicular to $\mathbf{r}$. The second term is the usual centripetal term. Then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

and

$$\begin{aligned}
\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} &= \vec{\mathbf{r}} \times m\left(\vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}\right) \\
&= m\left(\vec{\alpha} r^2 - \vec{\mathbf{r}}(\vec{\alpha} \cdot \vec{\mathbf{r}})\right) \\
&= m r^2 \vec{\alpha} = I\vec{\alpha}
\end{aligned}$$

since $\vec{\alpha}$ is perpendicular to $\mathbf{r}$, and for a particle $I = mr^2$.

$$A\hat{\mathbf{v}} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and finally:

$$A\hat{\mathbf{w}} = \frac{1}{18}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. Show that the vectors $\vec{\mathbf{u}} =$ (15, 12, 16), $\vec{\mathbf{v}} =$ (-20, 9, 12) and $\vec{\mathbf{w}} =$ (0,-4, 3) are mutually orthogonal and right handed. Determine the transformation matrix that transforms from the original (x, y, z) cordinate system, to a system with x' $-$axis along $\vec{\mathbf{u}}$, y' $-$axis along $\vec{\mathbf{v}}$ and z' $-$axis along $\vec{\mathbf{w}}$. Apply the transformation to find components of the vectors $\vec{\mathbf{a}} = (1, 1, 1)$, $\vec{\mathbf{b}} = (3, 2, 1)$ and $\vec{\mathbf{c}} = (-2, 1, -2)$ in the prime system. Discuss the result for vector $\vec{\mathbf{c}}$.

Two vectors are orthogonal if their dot product is zero.

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = -300 + 108 + 192 = 0$$

and

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -36 + 36 = 0$$

Finally

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -48 + 48 = 0$$

So the vectors are mutually orthogonal. In addition

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \times \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = \begin{pmatrix} 0 & -500 & 375 \end{pmatrix}$$
$$= 125\vec{\mathbf{w}}$$

So the vectors form a right-handed set.

To find the transformation matrix, first we find the magnitude of each vector and the corresponding unit vectors.

$$|\vec{\mathbf{u}}|^2 = 15^2 + 12^2 + 16^2 = 625 \Rightarrow |\vec{\mathbf{u}}| = \sqrt{625} = 25$$

So

$$\hat{\mathbf{u}} = \frac{1}{25}\begin{pmatrix} 15, & 12, & 16 \end{pmatrix}$$

$$\mathbb{A}_2\mathbb{A}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}$$

The new components of the orignal $z-$axis are:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

8. Does the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represent a rotation of the coordinate axes? If not, what transformation does it represent? Draw a diagram showing the old and new coordinate axes, and comment.

The determinant of this matrix is:

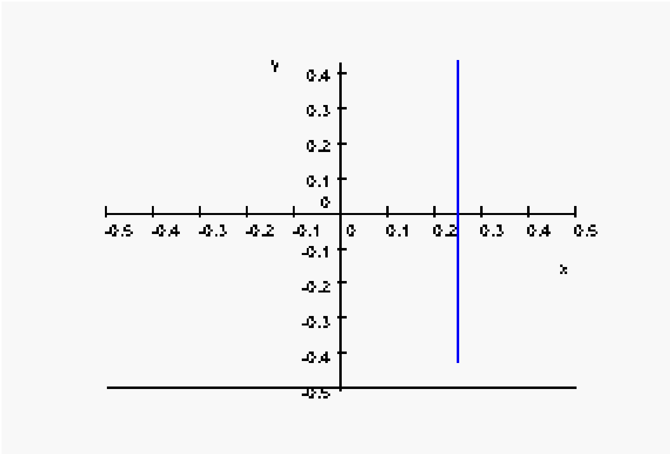
$$\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\cos^2\theta - \sin^2\theta = -1$$

Thus this transformation cannot be a rotation since a rotation matrix has determinant $+1$. Let's see where the axes go:

$$\mathbb{A}\hat{\mathbf{x}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

and

$$\mathbb{A}\hat{\mathbf{y}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$



The line in the $w-$plane

(b) The circle is

$$\begin{aligned}
[z-(2+i)][z-(2+i)]^* &= 1 \\
\left(\frac{1}{w}-i\right)\left(\frac{1}{w^*}+i\right) &= 1 \\
(1-iw)(1+iw^*) &= ww^* \\
1-i(w-w^*) &= 0 \\
1 &= i(2iv) \\
v &= -\frac{1}{2}
\end{aligned}$$

This is a straight line parallel to the $u-$axis. It extends from $w = \frac{1}{-1+i} = -\frac{1}{2} - \frac{i}{2}$ to $w = \frac{1}{1+i} = \frac{1}{2} - \frac{i}{2}$.

42. Show that $\Gamma(x) < 0$ for $-1 < x < 0.$

If $-1 < x < 0,$ then we can write

$$\Gamma(x) = \frac{\pi}{\sin \pi x \Gamma(1-x)} = \frac{\pi}{-|\sin \pi x|\Gamma(y)}$$

where $1 < y < 2.$ Then $\Gamma(y)$ is positive and hence $\Gamma(x)$ is negative.

43. Prove *Cauchy's inequality*: If $f(z)$ is analytic and bounded in a region R : $|z-z_0| < R,$ and $|f(z)| < M$ on the circle $|z-z_0| = r < R,$ then the coefficients in the Taylor series expansion of f about z_0 $\left(\text{eqn } 44\right)$ satisfy the inequality

Chapter 4: Fourier Series

1. Show that the Fourier series (equation 4.1) for a function $f(x)$ may be written:

$$f(x) = \sum_{n=0}^{\infty} k_n \cos(nx + \phi_n)$$

and find expressions for k_n and ϕ_n.

Expand the cosine to obtain:

$$f(x) = \sum_{n=0}^{\infty} k_n (\cos nx \cos \phi_n - \sin nx \sin \phi_n)$$

Comparing with equation 4.1, we have:

$$a_n = -k_n \sin \phi_n; \quad b_n = k_n \cos \phi_n$$

Thus

$$k_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\tan \phi_n = -\frac{a_n}{b_n}$$

where a_n and b_n are given by equations 4.7 and 4.8.

We may also work from the exponential series:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} r_n e^{i\theta_n} e^{inx} \\ &= \sum_{n=-\infty}^{\infty} r_n [\cos(nx + \theta_n) + i \sin(nx + \theta_n)] \end{aligned}$$

In this formulation c_n may be complex, but r_n is real. Thus if $f(x)$ is real, the imaginary terms must combine to give zero, leaving:

$$f(x) = \sum_{n=-\infty}^{\infty} r_n \cos(nx + \theta_n)$$

where

$$r_n = |c_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \right|$$

and

$$\phi_n = \theta_n = \arg(c_n) = \arg\left\{ \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \right\}$$

Note that

$$\phi_{-n} = -\phi_n$$

and

$$r_n = r_{-n}$$

and so the sine terms sum to zero in pairs, as required, and the cosines combine to give a sum over positive n only.

14. Find the Laplace transform of $\delta(t-a)$. Express the inverse as an integral using equation 5.19 and demonstrate that this integral possesses the sifting property**.**

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}$$

Thus

$$\delta(t-a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-sa}e^{st}ds$$

So we check for the sifting property:

$$\begin{aligned}\int_{-\infty}^{+\infty} f(t)\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-sa}e^{st}dsdt &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{-\infty}^{+\infty} f(t)e^{st}dte^{-sa}ds \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{-sa}ds = f(a)\end{aligned}$$

as required.

15. A disk of radius a and mass M lies in the $x-y$ plane. Express the density in terms of delta functions

(a) in rectangular Cartesian coordinates

In Cartesian coordinates, the disk is located at $z = 0$. We have:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)$$

if $\sqrt{x^2+y^2} < a,$ and zero otherwise. We can also express this in terms of step functions:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)\Theta\left(a - \sqrt{x^2+y^2}\right)$$

(b) in cylindrical coordinates

In cylindrical coordinates, the step function looks nicer:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)\Theta(a-\rho)$$

(c) in spherical coordinates.

The disk is at $\theta = \pi/2,$ so the density looks like:

	1	-1	i	j	k	$-i$	$-j$	$-k$
1	1	-1	i	j	k	$-i$	$-j$	$-k$
-1	-1	1	$-i$	$-j$	$-k$	i	j	k
i	i	$-i$	-1	k	$i(ij) = i^2 j = -j$	1	$-k$	j
j	j	$-j$	$-k$	-1	$j(-ji) = i$	$-k$	1	$-i$
k	k	$-k$	$(-ji)i = j$	$(ij)j = -i$	-1	$-j$	i	1
$-i$	$-i$	i	1	$-k$	j	-1	k	$-j$
$-j$	$-j$	j	k	1	$-i$	$-k$	-1	i
$-k$	$-k$	k	$-j$	i	1	j	$-i$	-1

Each element has an inverse: The identity 1 and also -1 each form their own inverse. The inverse of i is $-i,$ and so on for each of the elements.

(b) Show that i, j and k may be represented by the matrices:

$$i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 01 & 0 & 0 & 0 \end{pmatrix}$$

and

$$k = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Determine the classes of this group.

Let's check the matrix multiplication: