SOLUTIONS TO THE EXERCISES IN

LIKELIHOOD METHODS IN STATISTICS

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c. $V_\theta(Y) = -k''(\theta)$ and $V_\theta(X) = -\tilde{k}''(\theta)$. Hence, if $V_\theta(Y) = V_\theta(X)$ for all θ,

$$k(\theta) = \tilde{k}(\theta) + C_1\theta + C_2$$

where C_1 and C_2 are constants. It follows that

$$K_Y(t) = K_X(t) + C_1 t$$

and, hence, Y has the same distribution as $X + C_1$. For instance, if Y is normally distributed with mean θ and variance 1 and X is normally distributed with mean $\theta + 1$ and variance 1, then the densities of Y and X satisfy the conditions given but Y and X do not have the same distribution. ∎

1.3 Show that the natural parameter space of an exponential family model is convex.

Solution Let

$$\exp\{s(y)^T\eta - k(\eta) + D(y)\}$$

denote the density of the exponential family distribution. A parameter value η is in the natural parameter space provided that

$$\int \exp\{s(y)^T\eta + D(y)\}dy < \infty.$$

Suppose η_1 and η_2 are in the natural parameter space and let $0 < t < 1$. Then

$$\int \exp\{s(y)^T[t\eta_1 + (1-t)\eta_2] + D(y)\}dy = \int \exp\{ts(y)^T\eta_1 + (1-t)s(y)^T\eta_2 + D(y)\}dy.$$

Since the function $\exp(x)$ is convex

$$\begin{aligned}
&\int \exp\{ts(y)^T\eta_1 + (1-t)s(y)^T\eta_2 + D(y)\}dy \\
&\quad \le \int [t\exp\{s(y)^T\eta_1\} + (1-t)\exp\{s(y)^T\eta_2\}]\exp\{D(y)\}dy \\
&\quad \le t\int \exp\{s(y)^T\eta_1 + D(y)\}dy + (1-t)\int \exp\{s(y)^T\eta_2 + D(y)\}dy < \infty
\end{aligned}$$

so that $t\eta_1 + (1-t)\eta_2$ is in the natural parameter space. ∎

1.4 Let Y denote a nonnegative continuous random variable with density p. Suppose that Y is observed only if $Y \le y_o$ where y_o is a known constant.

a. Find the density function of Y given that Y is observed; denote this density by p_o.

b. Suppose that the density p is in the one-parameter exponential family. Under what conditions, if any, is p_o also in the one-parameter exponential family?

a. Find $\Pr(X_j = 0|X_{j-1} = 1)$, $\Pr(X_j = 1|X_{j-1} = 0)$, $\Pr(X_j = 0|X_{j-1} = 0)$.
b. Find the requirements on λ so that this describes a valid probability distribution for $X_1, \ldots, X_n$.
c. Show that $\Pr(X_j = x_j|X_{j-1} = x_{j-1})$, $j = 2, \ldots, n$, may be written

$$f(x_j, x_{j-1}) = \lambda^{x_j x_{j-1}}(1-\lambda)^{x_{j-1}(1-x_j}[\frac{(1-\lambda)\phi}{(1-\phi)}]^{(1-x_{j-1})x_j}[\frac{(1-2\phi+\lambda\phi)}{(1-\phi)}]^{(1-x_{j-1})(1-x_j)}$$

for $x_{j-1} = 0, 1$ and $x_j = 0, 1$.
d. Suppose that ϕ and λ are unknown parameters. Find a three-dimensional sufficient statistic for the model.

Solution
a. Since X_j only takes the values 0 and 1,

$$Pr(X_j = 0|X_{j-1} = 1) = 1 - Pr(X_j = 1|X_{j-1} = 1) = 1 - \lambda.$$

Since

$$\begin{aligned}&Pr(X_j = 1)\\ &\quad = Pr(X_j = 1|X_{j-1} = 1)Pr(X_{j-1} = 1) + Pr(X_j = 1|X_{j-1} = 0)Pr(X_{j-1} = 0)\\ &\quad = \lambda\phi + Pr(X_j = 1|X_{j-1} = 0)(1-\phi)\end{aligned}$$

it follows that

$$Pr(X_j = 1|X_{j-1} = 0) = \frac{\phi}{1-\phi}(1-\lambda)$$

and, hence, that

$$Pr(X_j = 0|X_{j-1} = 0) = 1 - \frac{\phi}{1-\phi}(1-\lambda) = \frac{1-2\phi+\phi\lambda}{1-\phi}.$$

b. Given that $0 < \phi < 1$, necessary and sufficient conditions for this to describe a valid probability distribution for $X_1, \ldots, X_n$ are that $Pr(X_j = 1|X_{j-1} = 1)$ and $Pr(X_j = 1|X_{j-1} = 0$ are in the interval $[0, 1]$. Hence, we must have $0 \le \lambda \le 1$ and

$$\lambda \ge 1 - \frac{1-\phi}{\phi} = \frac{2\phi-1}{\phi};$$

that is, λ must satisfy

$$\max\{0, \frac{2\phi-1}{\phi}\} \le \lambda \le 1.$$

c. This is easily verified by direct calculation.