Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices
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$$\frac{\partial \psi}{\partial z} = r u_r \,. \tag{1.84}$$

Substituting this in, find

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial \psi}{\partial z} + \frac{\partial}{\partial z}u_z = 0\,. \tag{1.85}$$

Rearrange to get

$$\frac{\partial}{\partial z}u_z = -\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial z}\psi \tag{1.86}$$

and then

$$\frac{\partial}{\partial z}u_z = \frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial}{\partial r}\psi\right)\,. \tag{1.87}$$

From this, we obtain

$$\frac{\partial \psi}{\partial r} = -r u_z \,, \tag{1.88}$$

so we find that

$$\boxed{\frac{\partial \psi}{\partial z} = r u_r} \tag{1.89}$$

$$\boxed{\frac{\partial \psi}{\partial r} = -r u_z} \tag{1.90}$$

satisfies the stream function requirements. Also, both of these relations could be multiplied by any constant, and conservation of mass would still be satisfied.

1.7 Consider the following two velocity gradient tensors:

(a) $\nabla \vec{u} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,.$

(b) $\nabla \vec{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,.$

Draw the streamlines for each velocity gradient tensor. With respect to the coordinate axes, identify which of these exhibits extensional strain and which exhibits shear strain. Following this, redraw the streamlines for each on axes that have been rotated 45° counterclockwise, using $x' = x/\sqrt{2} + y/\sqrt{2}$ and $y' = -x/\sqrt{2} + y/\sqrt{2}$. Are your conclusions about extensional and shear strain the same for the flow once you have rotated the axes? Do the definitions of extensional and shear strain depend on the coordinate system?

$$\eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial z}\frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 v}{\partial x^2}\right) + 2\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial z}\frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{\partial}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} \tag{1.105}$$

from equality of mixed partials and incompressibility, can subtract away half of the terms:

$$\eta \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} \tag{1.106}$$

1.13 For a one dimensional flow given by $\vec{\boldsymbol{u}} = u(y)$, the strain rate magnitude is given by $\frac{1}{2}\frac{\partial u}{\partial y}$ and the vorticity magnitude is given by $\frac{\partial u}{\partial y}$. Are the strain rate and vorticity proportional to each other in general? If not, why are they proportional in this case?

Solution:

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the solution for this problem is not available
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1.14 Write out the Navier–Stokes equations in cylindrical coordinates (see Appendix D). Simplify these equations for the case of plane symmetry.

Solution:

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the solution for this problem is not available
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1.15 Write out the Navier–Stokes equations in cylindrical coordinates (see Appendix D). Simplify these equations for the case of axial symmetry.

Solution:

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the solution for this problem is not available
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1.16 Write out the Navier–Stokes equations in spherical coordinates (see Appendix D). Simplify these equations for the case of axial symmetry.

Figure 1.32: Deformation of a grid by a stagnation flow, $\psi = Axy$. In general, grid spacing will be closer near origin and larger far from origin; this is not terribly clear from the sketch.

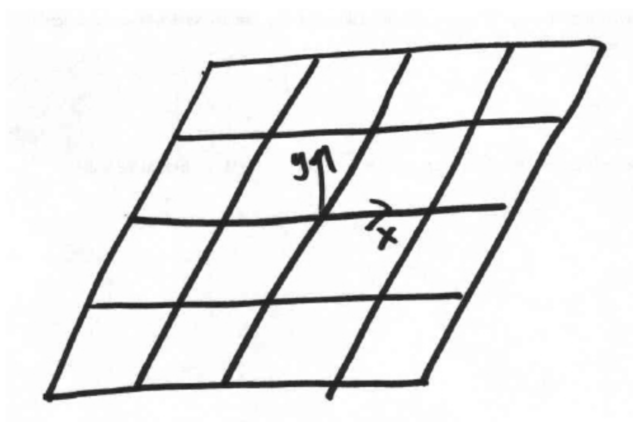

Figure 1.33: Deformation of a grid by a pure shear flow, $\psi = \frac{1}{2}By^2$.

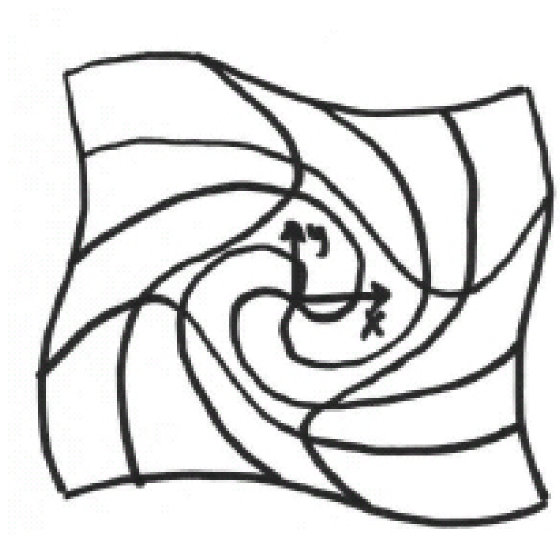

Figure 1.34: Deformation of a grid by an irrotational vortex, $\psi = C\ln\left(\sqrt{x^2+y^2}\right)$.

Solution:

complex elec displacement:

$$\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega = \underset{\sim}{\vec{\boldsymbol{D}}} + \frac{\underset{\sim}{\vec{\boldsymbol{J}}}}{j\omega} \tag{5.113}$$

$$\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega = \varepsilon\underset{\sim}{\vec{\boldsymbol{E}}} + \frac{\sigma\underset{\sim}{\vec{\boldsymbol{E}}}}{j\omega} \tag{5.114}$$

$$\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega = \left(\varepsilon + \frac{\sigma}{j\omega}\right)\underset{\sim}{\vec{\boldsymbol{E}}} \tag{5.115}$$

$$\boxed{\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega = \underset{\sim}{\varepsilon}\underset{\sim}{\vec{\boldsymbol{E}}}} \tag{5.116}$$

This complex electric displacement is a sum of two quantities: (1) the electric displacement and (2) the ohmic current, offset by a $\pi/2$ phase lag and normalized by ω to give it units of displacement rather than current.

divergence:

$$\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = \nabla\cdot\underset{\sim}{\vec{\boldsymbol{D}}} + \frac{\nabla\cdot\underset{\sim}{\vec{\boldsymbol{J}}}}{j\omega} \tag{5.117}$$

$$\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = \nabla\cdot\underset{\sim}{\vec{\boldsymbol{D}}} - \frac{1}{j\omega}\frac{\partial\underset{\sim}{\rho_{\mathrm{E}}}}{\partial t} \tag{5.118}$$

$$\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = \nabla\cdot\underset{\sim}{\vec{\boldsymbol{D}}} - \frac{j\omega}{j\omega}\underset{\sim}{\rho_{\mathrm{E}}} \tag{5.119}$$

$$\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = \nabla\cdot\underset{\sim}{\vec{\boldsymbol{D}}} - \underset{\sim}{\rho_{\mathrm{E}}} \tag{5.120}$$

$$\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = \rho_{\mathrm{E}} - \rho_{\mathrm{E}} \tag{5.121}$$

$$\boxed{\nabla\cdot\left(\underset{\sim}{\vec{\boldsymbol{D}}} + \underset{\sim}{\vec{\boldsymbol{J}}}/j\omega\right) = 0\,.} \tag{5.122}$$

Thus, if we combine the electric displacement with the ohmic current with the $1/\omega$ normalization and the $\pi/2$ phase offset, the divergence of the sum of these two quantities is zero. So $\nabla\cdot\underset{\sim}{\varepsilon}\underset{\sim}{\vec{\boldsymbol{E}}} = 0$. Thus the complex electric displacement obeys a simpler law than the electric displacement or the ohmic current.

Note that this applies *only* if both charge density and electric field are sinusoidal. The complex representations are valid because the equations are linear.

8.6 Given particles of radius a and density ρ_p and a fluid with viscosity $\eta = 1 \times 10^{-3}$ Pa s and density $\rho_w = 1$ kg/m^3, derive a relation for the terminal velocity of the particles when subjected to Earth's gravitational accleration (9.8 m/s^2). If a 20-μm-diameter microchannel has an axis aligned normal to the gravitational axis and is filled with a suspension of these particles, derive a relation for the time it would take for these particles to settle to the bottom of the channel. Calculate the velocity and settling time for

(a) 10-nm-diameter polystyrene beads ($\rho_p = 1300$ kg/m^3),
(b) 1-μm-diameter polystyrene beads ($\rho_p = 1300$ kg/m^3),
(c) 2.5-μm-diameter polystyrene beads ($\rho_p = 1300$ kg/m^3),
(d) human leukocytes (12 μm in diameter; $\rho_p = 1100$ kg/m^3).

Solution: The force on a particle is given by

$$F = \frac{4}{3}\pi a^3 \left(\rho_p - \rho_w\right) = \frac{1}{6}\pi d^3 \left(\rho_p - \rho_w\right) . \tag{8.75}$$

The viscous force on a particle is related to the velocity U by

$$F = 6\pi U \eta a = 3\pi U \eta d . \tag{8.76}$$

Setting these equal at the terminal velocity, we have

$$3\pi U \eta d = \frac{1}{6}\pi d^3 \left(\rho_p - \rho_w\right) . \tag{8.77}$$

Solving for U, we get an expression for the terminal velocity:

$$\boxed{U = \frac{1}{18\eta} d^2 \left(\rho_p - \rho_w\right) .} \tag{8.78}$$

The particles can be assumed to instantaneously achieve terminal velocity—this can be confirmed by calculating the gravitational force to the particle mass to determine the acceleration and estimating the time to equilibrate by dividing the acceleration by the terminal velocity. Thus the time to settle is given by the distance (20 μm) divided by the terminal velocity.

This can then be calculated for the four particles.

8.7 Consider flow through a shallow microfabricated channel of uniform depth d with a sharp 90° turn whose inside corner is located at the origin, depicted in Fig. 8.7. Assume that the flow far from the corner is uniform across the width of the channel, and assume that the channel can be approximated as being wide, so that the flow

11.3 Given the electrophoretic mobilities in Table 11.1 and the definition of molar conductivity in Eq. (11.24), calculate the molar conductivities of the following ions in water:

(a) H^+,
(b) OH^-,
(c) Li^+,
(d) SO_4^{-2}.

Solution:

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the solution for this problem is not available
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11.4 Consider the distribution of an ion of valence z in a 1D potential field $\varphi(y)$. Derive the Einstein relation by

- writing the equilibrium distribution c(φ),
- writing the 1D Nernst–Planck equations for ion transport in the y direction, and
- showing that the zero-flux condition at equilibrium requires that the Einstein relation hold.

Solution:

write $c(\varphi)$. In terms of φ, the concentration is

$$c = c_\infty \exp\left[\frac{-zF\varphi}{RT}\right]. \tag{11.31}$$

write 1D Nernst–Planck eqs. The steady 1D Nernst–Planck equation is

$$\frac{\partial c}{\partial t} = 0 = -\frac{\partial}{\partial y}\left[-D\frac{\partial c}{\partial y} - \mu_{\mathrm{EP}} c \frac{\partial \varphi}{\partial y}\right]. \tag{11.32}$$

We evaluate the first term by taking the derivative of c:

$$-D\frac{\partial c}{\partial y} = -D\frac{-zF}{RT}\frac{\partial \varphi}{\partial y} c_\infty \exp\left[\frac{-zF\varphi}{RT}\right]. \tag{11.33}$$

We expand the second term by writing out the expression for c:

$$-\mu_{\mathrm{EP}} c \frac{\partial \varphi}{\partial y} = -\mu_{\mathrm{EP}}\frac{\partial \varphi}{\partial y} c_\infty \exp\left[\frac{-zF\varphi}{RT}\right]. \tag{11.34}$$

The description of water models here is restricted to static charge models and ignores charge-on-spring models, inducible dipole models, and fluctuating charge models, as well as any aspect of quantum treatment of the water molecule. Sources that discuss water models, a description of water properties, or the relation between these include [304, 305, 306, 307].

H.7 Exercises

H.1 Consider a pair potential given by

$$e_2 = \infty \quad \text{if} \quad \Delta r < a \tag{H.29}$$

$$e_2 = k_B T \quad \text{if} \quad a < \Delta r < 2a \tag{H.30}$$

$$e_2 = 0 \quad \text{if} \quad 2a < \Delta r \tag{H.31}$$

Calculate and plot the Mayer f function f_M for this potential.

Solution:

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the solution for this problem is not available
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H.2 Write a numerical routine to solve the Ornstein–Zernike equation with hypernetted-chain closure to find the radial distribution function for a homogeneous Lennard–Jones fluid.

Proceed as follows:

(a) Use an iterative technique that, in turn, uses the hypernetted-chain closure in Eq. (H.17) to solve for f_{tc} and the Ornstein–Zernike equation (H.16) to solve for f_{dc}.

(b) Start by setting $f_{tc}(\Delta r) = f_{dc}(\Delta r) = 0$ on a domain that ranges from $\Delta r = 0$ to $\Delta r = 512\sigma$.

(c) In each step, define a new f_{tc} by using the hypernetted-chain relation:

$$f_{tc,new}(\Delta r) = -1 + \exp\left[-e_1(\Delta r)/k_B T + f_{tc,old}(\Delta r) - f_{dc}(\Delta r)\right] . \tag{H.32}$$

Note that $e_1(\Delta r)$ in this case is the Lennard–Jones potential.

(d) In each step, define a new f_{dc} by Fourier-transforming f_{tc} and f_{dc}, applying the Fourier-transformed Ornstein–Zernike equation to get a new $\hat{f}_{dc}$, and inverse Fourier-transforming $\hat{f}_{dc}$ to get a new f_{dc}. We do this because the Fourier-transformed Ornstein–Zernike equation is much easier to deal with (the spatial integral becomes a product when Fourier transformed):

$$\hat{f}_{tc}(k) = \hat{f}_{dc}(k) + \rho \hat{f}_{dc}(k) \hat{f}_{tc}(k) . \tag{H.33}$$