Instructor's Solutions Manual

PARTIAL DIFFERENTIAL EQUATIONS

with FOURIER SERIES and BOUNDARY VALUE PROBLEMS

Second Edition

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Contents

Exercises 1.2

1. We have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

So

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 v}{\partial t \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} = -\frac{\partial^2 u}{\partial x^2}.$$

Assuming that $\frac{\partial^2 v}{\partial t \partial x} = \frac{\partial^2 v}{\partial x \partial t}$, it follows that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, which is the one dimensional wave equation with $c = 1$. A similar argument shows that v is a solution of the one dimensional wave equation.

2. (a) For the wave equation in u, the appropriate initial conditions are $u(x, 0) = f(x)$, as given, and $u_t(x, 0) = -v_x(x, 0) = h'(x)$. (b) For the wave equation in v, the appropriate initial conditions are $v(x, 0) = h(x)$, as given, and $v_t(x, 0) = -u_x(x, 0) = f'(x)$.

3. $u_{xx} = F''(x + ct) + G''(x + ct)$, $u_{tt} = c^2 F''(x + ct) + c^2 G(x - ct)$. So $u_{tt} = c^u_{xx}$, which is the wave equation.

4. (a) Using the chain rule in two dimensions:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\
&= \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2} \\
&= \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} - c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2} \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Substituting into the wave equation, it follows that

$$c^2\frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

(b) The last equation says that $\frac{\partial u}{\partial \beta}$ is constant in α. So

$$\frac{\partial u}{\partial \beta} = g(\beta)$$

where g is an arbitrary differentiable function.

(c) Integrating the equation in (b) with respect to β, we find that $u = G(\beta) + F(\alpha)$, where G is an antiderivative of g and F is a function of α only.

(d) Thus $u(x, t) = F(x + ct) + G(x - ct)$, which is the solution in Exercise 3.

5. (a) We have $u(x, t) = F(x + ct) + G(x - ct)$. To determine F and G, we use the initial data:

$$u(x, 0) = \frac{1}{1 + x^2} \quad \Rightarrow \quad F(x) + G(x) = \frac{1}{1 + x^2}; \quad (1)$$

Solutions to Exercises 2.1

1. (a) $\cos x$ has period 2π. (b) $\cos \pi x$ has period $T = \frac{2\pi}{\pi} = 2$. (c) $\cos \frac{2}{3}x$ has period $T = \frac{2\pi}{2/3} = 3\pi$. (d) $\cos x$ has period 2π, $\cos 2x$ has period π, 2π, 3π,:. A common period of $\cos x$ and $\cos 2x$ is 2π. So $\cos x + \cos 2x$ has period 2π.
2. (a) $\sin 7\pi x$ has period $T = \frac{2\pi}{7\pi} = 2/7$. (b) $\sin n\pi x$ has period $T = \frac{2\pi}{n\pi} = \frac{2}{n}$. Since any integer multiple of T is also a period, we see that 2 is also a period of $\sin n\pi x$. (c) $\cos mx$ has period $T = \frac{2\pi}{m}$. Since any integer multiple of T is also a period, we see that 2π is also a period of $\cos mx$. (d) $\sin x$ has period 2π, $\cos x$ has period 2π; $\cos x + \sin x$ so has period 2π. (e) Write $\sin^2 2x = \frac{1}{2} - \frac{\cos 4x}{2}$. The function $\cos 4x$ has period $T = \frac{2\pi}{4} = \frac{\pi}{2}$. So $\sin^2 2x$ has period $\frac{\pi}{2}$.

3. (a) The period is $T = 1$, so it suffices to describe f on an interval of length 1. From the graph, we have

$$f(x) = \begin{cases} 0 & \text{if } -\frac{1}{2} \le x < 0, \\ 1 & \text{if } 0 \le x < \frac{1}{2}. \end{cases}$$

For all other x, we have $f(x+1) = f(x)$.
(b) f is continuous for all $x \neq \frac{k}{2}$, where k is an integer. At the half-integers, $x = \frac{2k+1}{2}$, using the graph, we see that $\lim_{h\to x^+} f(h) = 0$ and $\lim_{h\to x^-} f(h) = 1$. At the integers, $x = k$, from the graph, we see that $\lim_{h\to x^+} f(h) = 1$ and $\lim_{h\to x^-} f(h) = 0$. The function is piecewise continuous.
(c) Since the function is piecewise constant, we have that $f'(x) = 0$ at all $x \neq \frac{k}{2}$, where k is an integer. It follows that $f'(x+) = 0$ and $f'(x-) = 0$ (Despite the fact that the derivative does not exist at these points; the left and right limits exist and are equal.)
4. The period is $T = 4$, so it suffices to describe f on an interval of length 4. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -2 \le x \le 0, \\ -x+1 & \text{if } 0 < x < 2. \end{cases}$$

For all other x, we have $f(x+4) = f(x)$. (b) The function is continuous at all x. (c) (c) The function is differentiable for all $x \neq 2k$, where k is an integer. Note that f' is also 4-periodic. We have

$$f'(x) = \begin{cases} 1 & \text{if } -2 < x \le 0, \\ -1 & \text{if } 0 < x < 2. \end{cases}$$

For all other $x \neq 2k$, we have $f(x+4) = f(x)$. If $x = 0, \pm 4, \pm 8, \dots$, we have $f'(x+) = 1$ and $f'(x-) = -1$. If $x = \pm 2, \pm 6, \pm 10, \dots$, we have $f'(x+) = -1$ and $f'(x-) = 1$.

5. This is the special case $p = \pi$ of Exercise 6(b).
6. (a) A common period is $2p$. (b) The orthogonality relations are

$$\int_{-p}^{p} \cos \frac{m\pi x}{p} \cos \frac{n\pi x}{p}\, dx = 0 \quad \text{if } m \neq n,\ m,\ n = 0, 1, 2, \dots;$$

$$\int_{-p}^{p} \sin \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = 0 \quad \text{if } m \neq n,\ m,\ n = 1, 2, \dots;$$

$$\int_{-p}^{p} \cos \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = 0 \quad \text{for all } m = 0, 1, 2, \dots,\ n = 1, 2, \dots.$$

These formulas are established by using various addition formulas for the cosine and sine. For example, to prove the first one, if $m \neq n$, then

$$\begin{aligned} &\int_{-p}^{p} \cos \frac{m\pi x}{p} \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{2}\int_{-p}^{p} \left[\cos \frac{(m+n)\pi x}{p} + \cos \frac{(m-n)\pi x}{p}\right] dx \\ &= \frac{1}{2}\left[\frac{p}{m+n)\pi} \sin \frac{(m+n)\pi x}{p} + \frac{p}{m-n)\pi} \sin \frac{(m-n)\pi x}{p}\right]\Bigg|_{-p}^{p} = 0. \end{aligned}$$

Hence

$$f(x) = \frac{1}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\left[-\frac{\cos[(2k+1)\pi x]}{\pi(2k+1)^2} + \frac{\sin[(2k+1)\pi x]}{2k+1}\right].$$

22. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \begin{cases} \frac{x}{2}+1 & \text{if } -1 < x < 0, \\ 1-\frac{x}{2} & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{x}{2} \quad (-1 < x < 1).$$

As expected, $f(x) = f_e(x) + f_o(x)$. Let $g(x)$ be the function in Example 2 with $p = 1$ and $a = 1/2$. Then $f_e(x) = g(x) + 1/2$. So from Example 2 with $p = 1$ and $a = 1/2$, we obtain

$$\begin{aligned} f_e(x) &= \frac{1}{2} + \frac{1}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] \\ &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x]. \end{aligned}$$

From Exercise 2 with $p = 1$,

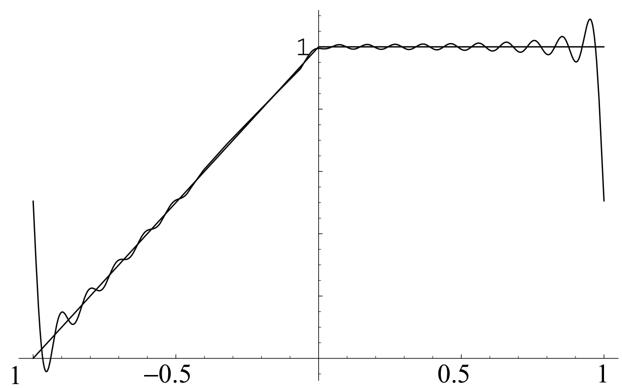
$$f_o(x) = \frac{1}{2}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

Hence

$$\begin{aligned} f(x) &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) \\ &= \frac{3}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n}\sin(n\pi x). \end{aligned}$$

Let's illustrate the convergence of the Fourier series. (This is one way to check that our answer is correct.)

```
f[x_] = Which[x < -1, 0, -1 < x < 0, x + 1, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 3 / 4 + 1 / Pi Sum[(1 - (-1) ^k) / (Pi k^2) Cos[ k Pi x] +
        (-1) ^ (k + 1) / k Sin[k Pi x], {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```



Sine series:

$$b_n = \frac{2}{p}\int_a^b \sin\frac{n\pi x}{p}\,dx = \frac{2}{p}\frac{p}{n\pi}\left(\cos\frac{n\pi a}{p} - \cos\frac{n\pi b}{p}\right);$$

thus the odd extension has the sine series

$$f_o(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi a}{p} - \cos\frac{n\pi b}{p}\right)\sin\frac{n\pi x}{p}.$$

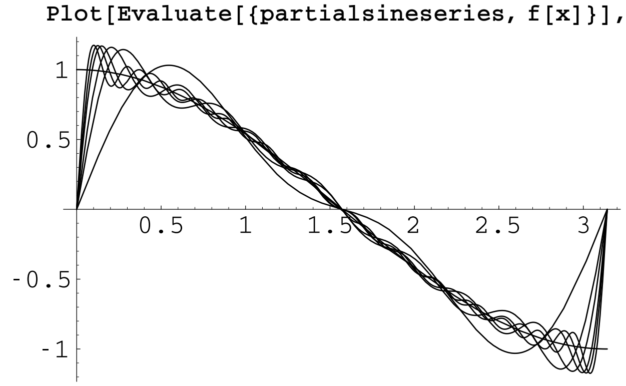
6. The even extension is the function $f_1(x) = \cos x$ for all x. Hence the Fourier series expansion is just $\cos x$. For the odd extension, we have, for $n > 1$,

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\,dx \\
&= \frac{2}{\pi}\left[\frac{\cos(1-n)x}{2(1-n)} - \frac{\cos(1+n)x}{2(1+n)}\right]\Big|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{(-1)^{n-1}}{(1-n)} - \frac{(-1)^{n+1}}{(1+n)} - \frac{1}{(1-n)} + \frac{1}{(1+n)}\right] \\
&= \frac{2n}{\pi}\frac{1+(-1)^n}{n^2-1}.
\end{aligned}$$

For $n = 1$, you can easily show that $b_1 = 0$. Thus the sine Fourier series is

$$\frac{2}{\pi}\sum_{n=2}^{\infty} n\frac{1+(-1)^n}{n^2-1}\sin nx = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{k}{(2k)^2-1}\sin(2kx).$$

```
b[k_] = 2 k / Pi (1 + (-1) ^k) / (k^2 - 1);
ss[n_, x_] :=  Sum[b[k] Sin[k x], {k, 2, n}];
partialsineseries = Table[ss[n, x], {n, 5, 30, 5}];
f[x_] = Cos[x]
Plot[Evaluate[{partialsineseries, f[x]}], {x, 0, Pi}]
```



7. The even extension is the function $|\cos x|$. This is easily seen by plotting the graph. The cosine series is (Exercise 8, Section 2.2):

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2-1}\cos(2nx).$$

Hence the steady-state solution is

$$y_p = -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t.$$

(b) We have

$$\begin{aligned}
y_p &= -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t,\\
(y_p)' &= \frac{1}{8}\sin t + \frac{1}{8}\cos t - \frac{8}{65}\sin 2t - \frac{1}{65}\cos 2t,\\
(y_p)'' &= \frac{1}{8}\cos t - \frac{1}{8}\sin t - \frac{16}{65}\cos 2t + \frac{2}{65}\sin 2t,\\
(y_p)'' + 4(y_p)' + 5y_p &= \left(\frac{1}{8} + \frac{4}{8} - \frac{5}{8}\right)\cos t + \left(-\frac{1}{8} + \frac{4}{8} + \frac{5}{8}\right)\sin t\\
&\quad + \left(\frac{2}{65} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t + \left(-\frac{16}{65} - \frac{4}{65} + \frac{20}{65}\right)\cos 2t\\
&= \sin t + \left(\frac{2}{75} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t - \frac{1}{2}\sin 2t,
\end{aligned}$$

which shows that y_p is a solution of the nonhomogeneous differential equation.

9. (a) Natural frequency of the spring is

$$\omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{10.1} \approx 3.164.$$

(b) The normal modes have the same frequency as the corresponding components of driving force, in the following sense. Write the driving force as a Fourier series $F(t) = a_0 + \sum_{n=1}^{\infty} f_n(t)$ (see (5). The normal mode, $y_n(t)$, is the steady-state response of the system to $f_n(t)$. The normal mode y_n has the same frequency as f_n. In our case, F is 2π-periodic, and the frequencies of the normal modes are computed in Example 2. We have $\omega_{2m+1} = 2m+1$ (the n even, the normal mode is 0). Hence the frequencies of the first six nonzero normal modes are 1, 3, 5, 7, 9, and 11. The closest one to the natural frequency of the spring is $\omega_3 = 3$. Hence, it is expected that y_3 will dominate the steady-state motion of the spring.

13. According to the result of Exercise 11, we have to compute $y_3(t)$ and for this purpose, we apply Theorem 1. Recall that y_3 is the response to $f_3 = \frac{4}{3\pi}\sin 3t$, the component of the Fourier series of $F(t)$ that corresponds to $n = 3$. We have $a_3 = 0$, $b_3 = \frac{4}{3\pi}$, $\mu = 1$, $c = .05$, $k = 10.01$, $A_3 = 10.01 - 9 = 1.01$, $B_3 = 3(.05) = .15$,

$$\alpha_3 = \frac{-B_3 b_3}{A_3^2 + B_3^2} = \frac{-(.15)(4)/(3\pi)}{(1.01)^2 + (.15)^2} \approx -.0611 \quad \text{and} \quad \beta_3 = \frac{A_3 b_3}{A_3 + B_3^2} \approx .4111.$$

So

$$y_3 = -.0611\cos 3t + .4111\sin 3t.$$

The amplitude of y_3 is $\sqrt{.0611^2 + .4111^2} \approx .4156$.

17. (a) In order to eliminate the 3rd normal mode, y_3, from the steady-state solution, we should cancel out the component of F that is causing it. That is, we must remove $f_3(t) = \frac{4\sin 3t}{3\pi}$. Thus subtract $\frac{4\sin 3t}{3\pi}$ from the input function. The modified input function is

$$F(t) - \frac{4\sin 3t}{3\pi}.$$

Its Fourier series is he same as the one of F, without the 3rd component, $f_3(t)$. So the Fourier series of the modified input function is

$$\frac{4}{\pi}\sin t + \frac{4}{\pi}\sum_{m=2}^{\infty}\frac{\sin(2m+1)t}{2m+1}.$$

series that we found in Exercise 8, Section 2.4. We have, for $0 < x < \pi$,

$$x \sin x = \frac{\pi}{2} \sin x - \frac{4}{\pi} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2-1)^2} \sin nx.$$

Let $x = \pi t$. Then, for $0 < t < 1$,

$$\pi t \sin \pi t = \frac{\pi}{2} \sin \pi t - \frac{4}{\pi} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2-1)^2} \sin n\pi t.$$

Equivalently, for $0 < x < 1$,

$$x \sin \pi x = \frac{1}{2} \sin \pi x - \frac{4}{\pi^2} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2-1)^2} \sin n\pi x.$$

So

$$b_1 = \frac{1}{2} \quad \text{and} \quad b_n = [1 + (-1)^n] \frac{-4n}{\pi^2(n^2-1)^2} \; (n \geq 2).$$

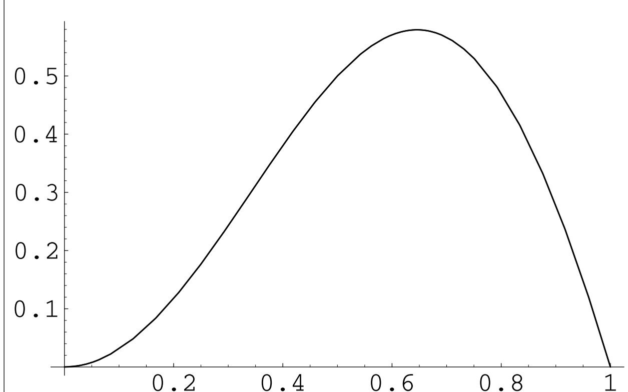
Thus

$$\begin{aligned} u(x,t) &= \frac{\sin \pi x \cos t}{2} - \frac{4}{\pi^2} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2-1)^2} \sin(n\pi x) \cos(nt) \\ &= \frac{\sin \pi x \cos t}{2} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)^2} \sin(2n\pi x) \cos(2nt). \end{aligned}$$

(b) Here is the initial shape of the string.

```
Clear[partsum, n, t, f]
Clear[f]
f[x_] = x Sin[Pi x]
partsum[x_, t_] =
  Sin[Pi x] Cos[t] / 2 - 16 / Pi^2 Sum[ Sin[2 n Pi x] Cos[2 n t] n / (4 n^2 - 1) ^2
    , {n, 1, 10}];
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}, PlotRange → All]

x Sin[π x]
```



9. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left(b_n \cos(n\pi t) + b_n^* \sin(n\pi t)\right),$$

Solutions to Exercises 7.4

1. Repeat the solution of Example 1 making some adjustments: $c = \frac{1}{2}$, $g_t(x) = \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{x^2}{t}}$,

$$\begin{aligned}
u(x, t) &= f * g_t(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{(x-s)^2}{t}}\, ds \\
&= \frac{20}{\sqrt{t\pi}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{t}}\, ds \quad (v = \frac{x-s}{\sqrt{t}},\ dv = -\frac{1}{\sqrt{t}}\, ds) \\
&= \frac{20}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{t}}}^{\frac{x+1}{\sqrt{t}}} e^{-v^2}\, ds \\
&= 10 \left(\operatorname{erf}(\frac{x+1}{\sqrt{t}}) - \operatorname{erf}(\frac{x-1}{\sqrt{t}}) \right).
\end{aligned}$$

3. Let us use an approach similar to Example 2. Fourier transform the boundary value problem and get:

$$\begin{aligned}
\frac{d}{dt}\widehat{u}(w, t) &= -w^2 \widehat{u}(w, t) \\
\widehat{u}(w, 0) &= \mathcal{F}(70 e^{-\frac{x^2}{2}}) = 70 e^{-\frac{w^2}{2}}.
\end{aligned}$$

Solve the equation in $\widehat{u}$:

$$\widehat{u}(w, t) = A(w) e^{-w^2 t}.$$

Apply the boundary condition:

$$\widehat{u}(w, t) = 70 e^{-\frac{w^2}{2}}) e^{-w^2 t} = 70 e^{-w^2 (t+\frac{1}{2})}.$$

Inverse Fourier transform:

$$\begin{aligned}
u(x, t) &= \mathcal{F}^{-1}\left(70 e^{-w^2(t+\frac{1}{2})}\right) \qquad (\frac{1}{2a} = t + \frac{1}{2}) \\
&= \frac{70}{\sqrt{2t+1}} \mathcal{F}^{-1}\left(\sqrt{2t+1}\, e^{-\frac{w^2}{2a}}\right) \qquad (a = \frac{1}{2t+1}) \\
&= \frac{70}{\sqrt{2t+1}} e^{-\frac{x^2}{2(2t+1)}},
\end{aligned}$$

where we have used Theorem 5, Sec. 7.2.

5. Apply (4) with $f(s) = s^2$:

$$\begin{aligned}
u(x, t) &= f * g_t(x) \\
&= \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^2 e^{-\frac{(x-s)^2}{t}}\, ds.
\end{aligned}$$

You can evaluate this integral by using integration by parts twice and then appealing to Theorem 5, Section 7.2. However, we will use a different technique based on the operational properties of the Fourier transform that enables us to evaluate a much more general integral. Let n be a nonnegative integer and suppose that f and $s^n f(s)$ are integrable and tend to 0 at $\pm\infty$. Then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^n f(s)\, ds = (i)^n \left[\frac{d^n}{dw^n} \mathcal{F}(f)(w) \right]_{w=0}.$$

This formula is immediate if we recall Theorem 3(ii), Section 7.2, and that

$$\widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s)\, ds.$$

49. $3\,y'' + 13\,y' + 10\,y = \sin x, \quad y_1 = e^{-x}$.

As in the previous exercise, let

$$y_1 = e^{-x},\ y = ve^{-x},\ y' = v'e^{-x} - ve^{-x},\ y'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}.$$

Then

$$\begin{aligned} 3\,y'' + 13\,y' + 10\,y = \sin x \quad &\Rightarrow \quad 3(v''e^{-x} - 2v'e^{-x} + ve^{-x}) \\ &\qquad +13(v'e^{-x} - ve^{-x}) + 10ve^{-x} = \sin x \\ &\Rightarrow \quad 3v'' + 7v' = e^{x}\sin x \\ &\Rightarrow \quad v'' + \frac{7}{3}v' = \frac{1}{3}e^{x}\sin x. \end{aligned}$$

We now solve the first order o.d.e. in v':

$$\begin{aligned} e^{7x/3}v'' + \frac{7}{3}e^{7x/3}v' &= e^{7x/3}\frac{1}{3}e^{x}\sin x \\ \frac{d}{dx}\left[e^{7x/3}v'\right] &= \frac{1}{3}e^{10x/3}\sin x \\ e^{7x/3}v' &= \frac{1}{3}\int \frac{1}{3}e^{10x/3}\sin x\,dx \\ &= \frac{1}{3}\frac{e^{10x/3}}{(\frac{10}{3})^2+1}(\frac{10}{3}\sin x - \cos) + C \\ v' &= \frac{e^{x}}{109}(10\sin x - \frac{9}{3}\cos) + C. \end{aligned}$$

(We used the table of integrals to evaluate the preceding integral. We will use it again below.) Integrating once more,

$$\begin{aligned} v &= \frac{10}{109}\int e^{x}\sin x\,dx - \frac{9}{327}\int e^{x}\cos x\,dx \\ &= \frac{10}{109}\frac{e^{x}}{2}(\sin x - \cos x) - \frac{9}{327}\frac{e^{x}}{2}(\cos x + \sin x) + C \\ y &= vy_1 = \frac{10}{218}(\sin x - \cos x) - \frac{9}{654}(\cos x + \sin x) + Ce^{-x} \\ &= -\frac{13}{218}\cos x + \frac{7}{218}\sin x + Ce^{-x}. \end{aligned}$$

50. $x\,y'' - (1+x)y' + y = x^3, \quad y_1 = e^{x}$. We have

$$y_1 = e^{x},\ y = ve^{x},\ y' = v'e^{x} + ve^{x},\ y'' = v''e^{x} + 2v'e^{x} + ve^{x}.$$