

# Student Solutions Manual

Phillip E. Bedient • Richard E. Bedient

Eighth Edition

# Elementary Differential Equations

Earl D. Rainville  
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# Chapter 1

## Definitions; Families of Curves

### 1.2 Definitions

All answers in this section are determined by inspection.

1. The equation is ordinary, linear in  $x$ , and of order 2.
3. The equation is ordinary, nonlinear, and of order 1.
5. The equation is ordinary, linear in  $y$ , and of order 3.
7. The equation is partial, linear in  $u$ , and of order 2.
9. The equation is ordinary, linear in  $x$  or  $y$ , and of order 2.
11. The equation is ordinary, linear in  $y$ , and of order 1.
13. The equation is ordinary, nonlinear, and of order 3.
15. The equation is ordinary, linear in  $y$ , and of order 2.

### 1.3 Families of Solutions

1. Rewriting the equation yields  $y = \int x^3 + 2x \, dx + c$ . Integrating, we have  $y = \frac{1}{4}x^4 + x^2 + c$ .
3. Rewriting the equation yields  $y = 4 \int \cos 6x \, dx + c$ . Integrating, we have  $y = \frac{2}{3} \sin 6x + c$ .
5. Rewriting the equation yields  $y = 2 \int \frac{1}{x^2 + 2^2} \, dx + c$ . Integrating, we have  $y = \arctan(x/2) + c$ .
7. Rewriting the equation yields  $y = 3 \int e^x \, dx$ . Integrating, we have  $y = 3e^x + c$ . Substituting the initial conditions gives  $6 = 3 + c$  or  $c = 3$  so  $y = 3e^x + 3$ .
9. As in Example 1.2,  $y = ce^{4x}$ . Substituting the initial conditions gives  $3 = ce^0 = c$  so  $y = 3e^{4x}$ .
11. Rewriting the equation yields  $y = 4 \int \sin 2x \, dx$ . Integrating, we have  $-2 \cos 2x + c$ . Substituting the initial conditions gives  $2 = -2 \cos \pi + c = 2 + c$  or  $c = 0$  so  $y = -2 \cos 2x$ .



## 7.3. THE AUXILIARY EQUATION: REPEATED ROOTS

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25. The auxiliary equation is  $m^2 - 2m - 3 = 0$  and its roots are  $m = 3, -1$ . The general solution is  $y = c_1 e^{3x} + c_2 e^{-x}$  and  $y' = 3c_1 e^{3x} - c_2 e^{-x}$ . But  $y(0) = c_1 + c_2 = 4$  and  $y'(0) = 3c_1 - c_2 = 0$ , so that  $c_1 = 1, c_2 = 3$ . The particular solution is  $y = e^{3x} + 3e^{-x}$ . Thus  $y(1) = e^3 + e^{-1}$ .
27. The auxiliary equation is  $m^2 - m - 6 = 0$  and its roots are  $m = 3, -2$ . The general solution is  $y = c_1 e^{3x} + c_2 e^{-2x}$  and  $y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$ . But  $y(0) = c_1 + c_2 = 3$  and  $y'(0) = 3c_1 - 2c_2 = -1$ , so that  $c_1 = 1, c_2 = 2$ . The particular solution is  $y = e^{3x} + 2e^{-2x}$ . Thus  $y(1) = e^3 + 2e^{-2}$ .
29. The auxiliary equation is  $m^3 - 2m^2 - 5m + 6 = 0$  and its roots are  $m = 1, 3, -2$ . The general solution is  $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$ , so that  $y' = c_1 e^x + 3c_2 e^{3x} - 2c_3 e^{-2x}$ , and  $y'' = c_1 e^x + 9c_2 e^{3x} + 4c_3 e^{-2x}$ . But  $y(0) = c_1 + c_2 + c_3 = 1, y'(0) = c_1 + 3c_2 - 2c_3 = -7$ , and  $y''(0) = c_1 + 9c_2 + 4c_3 = -1$ . Thus  $c_1 = 0, c_2 = -1, c_3 = 2$ . The particular solution is  $y = -e^{3x} + 2e^{-2x}$  and  $y(1) = -e^3 + 2e^{-2}$ .

## 7.3 The Auxiliary Equation: Repeated Roots

1. The auxiliary equation is  $m^2 - 6m + 9 = 0$  and its roots are  $m = 3, 3$ . The general solution is  $y = (c_1 + c_2 x)e^{3x}$ .
3. The auxiliary equation is  $4m^3 + 4m^2 + m = 0$  and its roots are  $m = 0, -\frac{1}{2}, -\frac{1}{2}$ . The general solution is  $y = c_1 + (c_2 + c_3 x)\exp(-\frac{1}{2}x)$ .
5. The auxiliary equation is  $m^4 + 6m^3 + 9m^2 = 0$  and its roots are  $m = 0, 0, -3, -3$ . The general solution is  $y = c_1 + c_2 x + (c_3 + c_4 x)e^{-3x}$ .
7. The auxiliary equation is  $4m^3 - 3m + 1 = 0$  and its roots are  $m = -1, \frac{1}{2}, \frac{1}{2}$ . The general solution is  $y = (c_1 + c_2 x)\exp(\frac{1}{2}x) + c_3 e^{-x}$ .
9. The auxiliary equation is  $m^3 + 3m^2 + 3m + 1 = 0$  and its roots are  $m = -1, -1, -1$ . The general solution is  $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$ .
11. The auxiliary equation is  $m^5 - m^3 = 0$  and its roots are  $m = 0, 0, 0, 1, -1$ . The general solution is  $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$ .
13. The auxiliary equation is  $4m^4 + 4m^3 - 3m^2 - 2m + 1 = 0$  and its roots are  $m = -1, -1, \frac{1}{2}, \frac{1}{2}$ . The general solution is  $y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)\exp(\frac{1}{2}x)$ .
15. The auxiliary equation is  $m^4 + 3m^3 - 6m^2 - 28m - 24 = 0$  and its roots are  $m = -2, -2, -2, 3$ . The general solution is  $y = (c_1 + c_2 x + c_3 x^2)e^{-2x} + c_4 e^{3x}$ .
17. The auxiliary equation is  $4m^5 - 23m^3 - 33m^2 - 17m - 3 = 0$  with roots  $m = -1, -1, 3, -\frac{1}{2}, -\frac{1}{2}$ . The general solution is  $y = (c_1 + c_2 x)e^{-x} + c_3 e^{3x} + (c_4 + c_5 x)\exp(-\frac{1}{2}x)$ .
19. The auxiliary equation is  $m^4 - 5m^2 - 6m - 2 = 0$  and its roots are  $m = 1 \pm \sqrt{3}, -1, -1$ . The general solution is  $y = (c_1 + c_2 x)e^{-x} + c_3 \exp[(1 + \sqrt{3})x] + c_4 \exp[(1 - \sqrt{3})x]$ .
21. The auxiliary equation is  $m^2 + 4m + 4 = 0$  and its roots are  $m = -2, -2$ . The general solution is  $y = (c_1 + c_2 x)e^{-2x}$  and  $y' = (-2c_1 + c_2 - 2c_2 x)e^{-2x}$ . But  $y(0) = c_1 = 1$  and  $y'(0) = -2c_1 + c_2 = -1$ , so that  $c_1 = c_2 = 1$ . The particular solution is  $y = (1 + x)e^{-2x}$ .

## 12.4. SIMPLE NETWORKS

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The first of these equations can be written  $\frac{dI_2}{dt} = -\frac{R_1}{L_2}I_1 - \frac{R_2}{L_2}I_2 + \frac{E}{L_2}$ . Differentiating the second equation and using the third equation to eliminate  $I_1$  yields

$$\begin{aligned} R_1 \frac{dI_1}{dt} + R_3 \frac{dI_3}{dt} + \frac{1}{C_3} I_3 &= 0, \\ R_1 \frac{dI_1}{dt} + R_3 \left( \frac{dI_1}{dt} - \frac{dI_2}{dt} \right) + \frac{1}{C_3} (I_1 - I_2) &= 0. \end{aligned}$$

Replacing  $\frac{dI_2}{dt}$  by its equivalent gives us

$$\begin{aligned} (R_1 + R_3) \frac{dI_1}{dt} - \frac{R_3}{L_2} (-R_1 I_1 - R_2 I_2 + E) + \frac{1}{C_3} (I_1 - I_2) &= 0, \\ (R_1 + R_3) \frac{dI_1}{dt} = \left( -\frac{R_1 R_3}{L_2} - \frac{1}{C_3} \right) I_1 + \left( -\frac{R_2 R_3}{L_2} + \frac{1}{C_3} \right) I_2 + \frac{R_3 E}{L_2}. \end{aligned}$$

The system in  $I_1$  and  $I_2$  can now be written

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}' = \begin{pmatrix} -\frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} & -\frac{C_3 R_2 R_3 - L_2}{C_3 L_2 (R_1 + R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} \frac{R_3 E}{L_2 (R_1 + R_3)} \\ \frac{E}{L_2} \end{pmatrix}.$$

The nature of the solutions of this system depend upon the roots of the characteristic equation

$$\begin{vmatrix} -\frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} - m & -\frac{C_3 R_2 R_3 - L_2}{C_3 L_2 (R_1 + R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} - m \end{vmatrix} = 0,$$

which may be written

$$m^2 + \left[ \frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} + \frac{R_2}{L_2} \right] m + \frac{R_2 (C_3 R_1 R_3 + L_2)}{C_3 L_2^2 (R_1 + R_3)} - \frac{R_1 (C_3 R_2 R_3 - L_2)}{C_3 L_2^2 (R_1 + R_3)} = 0,$$

or

$$C_3 L_2 (R_1 + R_3) m^2 + [C_3 (R_1 R_2 + R_2 R_3 + R_3 R_1) + L_2] m + R_1 + R_2 = 0.$$

Note that the answer given in the book has a typographic error. The last term should be  $R_2$ , not  $R_3$ .

18.11. MANY-TERM RECURRENCE RELATIONS

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The indicial equation is  $(c+1)^2 = 0$  and  $a_n = \frac{-a_{n-1}}{n+c+1}$  for  $n \geq 1$ . Solving this recurrence relation we get

$$\begin{aligned} a_n &= \frac{(-1)^n a_0}{(c+2) \cdots (c+n+1)}, \\ y_c &= x^c + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+c}}{(c+2) \cdots (c+n+1)}, \\ \frac{\partial y_c}{\partial c} &= y_c \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+c}}{(c+2) \cdots (c+n+1)} \left[ -\frac{1}{c+2} - \cdots - \frac{1}{c+n+1} \right]. \end{aligned}$$

Substituting  $c = -1$  gives the solutions

$$\begin{aligned} y_1 &= x^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{n!}, \\ y_2 &= y_1 \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n x^{n-1}}{n!}. \end{aligned}$$

25. Set  $L(y) = x(1-2x)y'' - 2(2+x)y' + 18y$ . Then

$$\begin{aligned} L(y) &= \sum_{n=0}^{\infty} [(n+c)(n+c-1) - 4(n+c)] a_n x^{n+c-1} \\ &\quad - \sum_{n=0}^{\infty} [2(n+c)(n+c-1) + 2(n+c) - 18] a_n x^{n+c} \\ &= \sum_{n=0}^{\infty} (n+c)(n+c-5) a_n x^{n+c-1} - \sum_{n=0}^{\infty} 2(n+c-3)(n+c+3) a_n x^{n+c} \\ &= c(c-5) a_0 x^{c-1} + \sum_{n=1}^{\infty} [(n+c)(n+c-5) a_n - 2(n+c-4)(n+c+2) a_{n-1}] x^{n+c-1}. \end{aligned}$$

Choosing  $c = 0$  and  $n(n-5)a_n = 2(n-4)(n+2)a_{n-1}$  for  $n \geq 1$ ,

$$\begin{aligned} 1 \cdot (-4) a_1 &= 2 \cdot (-3) \cdot 3 a_0, & a_1 &= \frac{9}{2} a_0, \\ 2 \cdot (-3) a_2 &= 2 \cdot (-2) \cdot 4 a_1, & a_2 &= 12 a_0, \\ 3 \cdot (-2) a_3 &= 2 \cdot (-1) \cdot 5 a_2, & a_3 &= 20 a_0, \\ 4 \cdot (-1) a_4 &= 2 \cdot (0) \cdot 6 a_3, & a_4 &= 0, \\ 5 \cdot (0) a_5 &= 2 \cdot (1) \cdot 7 a_4, & a_5 &\text{arbitrary.} \end{aligned}$$

This linear differential equation has general solution  $u(x, s) = c_1(s)e^{4sx} + c_2(s)e^{-4sx} - \frac{2}{s^2}$ . In order for  $\lim_{x \rightarrow \infty} u$  to exist we must take  $c_1(s) = 0$ . As  $x \rightarrow 0$  we need to take  $\frac{1}{s^2} = c_2(s) - \frac{2}{s^2}$ . That is  $c_2(s) = \frac{3}{s^2}$ . We therefore have  $u(x, s) = \frac{3}{s^2}e^{-4sx} - \frac{2}{s^2}$ . An inverse transform now yields the solution  $y(x, t) = 3(t - 4x)\alpha(t - 4x) - 2t$ .

## 25.2 The Wave Equation

1. Direct application of the Laplace transform gives us the transformed system

$$s^2u - s(x - x^2) = \frac{d^2u}{dx^2}, \quad x \rightarrow 0^+, u \rightarrow 0, \quad x \rightarrow 1^-, u \rightarrow 0.$$

The linear equation has as its general solution  $u = \frac{x}{s} - \frac{x^2}{s} - \frac{2}{s^3} + c_1e^{-sx} + c_2e^{sx}$ . The condition  $x \rightarrow 0^+, u \rightarrow 0$  implies that  $c_1 + c_2 = 2/s^3$ . The condition  $x \rightarrow 1^-, u \rightarrow 0$  implies that  $e^{-s}c_1 + e^sc_2 = 2/s^3$ . From these two equations we obtain

$$c_1 = \frac{2}{s^3(1 + e^{-s})} = \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^n e^{-ns} \quad \text{and} \quad c_2 = \frac{2e^{-s}}{s^3(1 + e^{-s})} = \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^{n+1} e^{-(n+1)s}.$$

Therefore

$$u(x, s) = \frac{x}{s} - \frac{x^2}{s} - \frac{2}{s^3} + \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^n e^{-(n+x)s} + \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^{n+1} e^{-(n+1-x)s}.$$

The inverse transform now yields

$$y(x, t) = x - x^2 - t^2 + \sum_{n=0}^{\infty} (-1)^n [(t - n - x)^2 \alpha(t - n - x) + (t - n - 1 + x)^2 \alpha(t - n - 1 + x)].$$

## 25.5 Diffusion in a Slab of Finite Width

1. Direct application of the Laplace transform gives us the transformed system

$$sw - 1 = \frac{d^2w}{dx^2}, \quad x \rightarrow 0^+, w \rightarrow 0, \quad x \rightarrow 1^-, \frac{dw}{dx} \rightarrow 0.$$

The linear equation has as its general solution  $w = \frac{1}{s} + c_1 \sinh(x\sqrt{s}) + c_2 \cosh(x\sqrt{s})$ . The condition  $x \rightarrow 0^+, w \rightarrow 0$  implies that  $c_2 = -1/s$ . Thus

$$\frac{dw}{dx} = \sqrt{s}c_1 \cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}} \sinh(x\sqrt{s}).$$