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BOUNDARY VALUE PROBLEMS AND PARTIAL DIFFERENTIAL EQUATIONS

Student Solutions Manual

DAVID L. POWERS

FIFTH EDITION

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Chapter 0

0.1 Homogeneous Linear Equations

1. You should be able to write out the solution without going through any algebra

$$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

3.(a) Treat this as a constant-coefficients equation. The characteristic equation is $m^2 = 0$, with double root $m = 0$. Therefore the solution of the differential equation is $u(t) = c_1 + c_2 t$.

(b) Because there is no u or du/dt term, you can integrate directly, twice: $du/dt = c_2$, $u = c_2 t + c_1$.

5. Do the indicated differentiation.

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} - \frac{\lambda^2}{r^2}W = 0.$$

This is a Cauchy-Euler equation. Guess $w = r^m$ so the characteristic equation is $m(m-1) + m - \lambda^2 = 0$ or $m^2 - \lambda^2 = 0$, with solutions $m = \lambda$, $m = -\lambda$. The general solution is

$$w(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}.$$

7. This differential equation is best solved by integrating (since there is no term in v).

$$(h + kx)\frac{dv}{dx} = c_1$$

$$\frac{dv}{dx} = \frac{c_1}{h + kx}$$

$$v = \frac{c_1}{k}\ln(h + kx) + c_2$$

9. Solve by integrating, since there is no term in u:

$$x^3\frac{du}{dx} = c_1; \quad \frac{du}{dx} = c_1 x^{-3};$$

$$u = -\frac{c_1}{2}x^{-2} + c_2.$$

11. Solve by integrating, since there is no term in u.

$$r\frac{du}{dr} = c_1; \quad \frac{du}{dr} = \frac{c_1}{r}; \quad u = c_1\ln(r) + c_2.$$

13. The characteristic polynomial is $(m^4 - \lambda^4) = (m^2 + \lambda^2)(m^2 - \lambda^2)$ with roots $m = \pm\lambda$, $\pm i\lambda$. The general solution of the differential equation is

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3\cos(\lambda x) + c_4\sin(\lambda x).$$

$$a_0 = \frac{1}{2}$$

by geometry; $a_n = 0$ because the extended function is $\frac{1}{2}$ plus an odd square wave of height $\frac{1}{2}$; then $b_n = (1 - \cos(n\pi))/n\pi$.

9. Note that the given function is the sum of the odd and even extensions of $f(x) = x$, $0 < x < a$. Use this fact to find $a_0 = a/2$, $a_n = 2a(\cos(n\pi) - 1)/(n\pi)^2$; $b_n = -2a\cos(n\pi)/(n\pi)$. The periodic function has discontinuities at $x = \pm a$, $\pm 3a$, etc. The table shows the sum of the series, determined by using the convergence theorem.

x	$-a$	$-a/2$	0	a	$2a$
sum	a	0	0	a	0

11. $a_0 = \frac{3}{4}$ by geometry.

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos(nx)dx + \int_{\pi/2}^{\pi}\frac{1}{2}\cos(nx)dx\right] = \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\Big|_0^{\pi/2} + \frac{\sin(nx)}{2n}\Big|_{\pi/2}^{\pi}\right] = \frac{\sin(\frac{n\pi}{2})}{n\pi}.$$

This function has discontinuities at $x = \pm\pi/2$, $\pm 3\pi/2$, etc. The convergence theorem gives the values in the table

x	0	$\pi/2$	π	$3\pi/2$	2π
sum	1	3/4	1/2	3/4	1

13. The odd periodic extension, period 2, accidentally has period 1 because of odd symmetry about the point $x = 1/2$. Discontinuities occur at $x = 0, \pm 1, \pm 2$, etc., and the sum of the series is 0 at those points.

15. The function is odd, continuous, and sectionally smooth, so $f(x) = \sum b_n \sin(nx)$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi/2}\sin(2x)\sin(nx)dx$$

$$= \frac{2}{\pi}\left[\frac{\sin((n-2)x)}{2(n-2)} - \frac{\sin((n+2)x)}{2(n+2)}\right|_0^{\pi/2} = -\frac{\sin(n\pi/2)}{\pi}\left[\frac{1}{n-2} - \frac{1}{n+2}\right], (n \neq 2).$$

Note that $\sin((n\pm 2)\pi/2) = \sin(n\pi/2 \pm \pi) = -\sin(n\pi/2)$. The integration formula would require division by 0 if $n = 2$, so b_2 must be found separately.

17. Use $\cos(nx) = \mathrm{Re}(e^{inx})$, then find the real part of $\sum_1^N e^{inx}$:

$$S = \sum_1^N e^{inx} = \frac{e^{i(N+1)x} - e^{ix}}{e^{ix} - 1}.$$

This can be reduced by standard algebra, but it is quicker to multiply numerator and denominator by $e^{-ix/2}$; so the sum becomes

$$S = \frac{e^{i(N+1/2)x} - e^{ix/2}}{e^{ix/2} - e^{-ix/2}}.$$

$$A_n \cosh(\mu_n a) = \frac{2}{b}\int_0^b Sy\cos(\mu_n y)dy = 2Sb\left[\frac{1}{(n-\frac{1}{2})^2\pi^2} + \frac{\sin((n-\frac{1}{2})\pi)}{(n-\frac{1}{2})\pi}\right] = c_n.$$

Finally,

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \frac{\cosh(\mu_n x)}{\cosh(\mu_n a)}\cos(\mu_n y).$$

7. The problem has homogeneous boundary conditions at $x = 0$ and $x = a$. The eigenvalue problem is

$$X'' + \lambda^2 X = 0, \quad 0 < x < a$$
$$X'(0) = 0, \quad X(a) = 0$$

with $X_n(x) = \cos(\lambda_n x)$, $\lambda_n = (n-\frac{1}{2})\pi/a$. Also $Y_n'' - \lambda_n^2 Y_n = 0$ and $Y_n'(0) = 0$, so $Y_n(y) = \cosh(\lambda_n y)$. Then

$$w(x,y) = \sum_{n=1}^{\infty} a_n \cosh(\lambda_n y)\cos(\lambda_n x).$$

The remaining condition, at $y = b$, is $w(x,b) = Sb(x-a)/a$, or

$$\sum_{n=1}^{\infty} a_n \cosh(\lambda_n b)\cos(\lambda_n x) = Sb(x-a)/a$$

$$a_n \cosh(\lambda_n b) = \frac{2}{a}\int_0^a \frac{Sb}{a}(x-a)\cos(\lambda_n x)dx = -\frac{2Sb}{(n-\frac{1}{2})^2\pi^2}.$$

9. This is a routine problem from Section 4.2. The solution has the form

$$w(x,y) = \sum_{n=1}^{\infty} \frac{c_n\sinh(\lambda_n y) + a_n\sinh(\lambda_n(b-y))}{\sinh(\lambda_n b)}\sin(\lambda_n x).$$

The coefficients are

$$a_n = c_n = \frac{2}{a}\int_0^a -\frac{Hx(a-x)}{2}\sin(\lambda_n x)dx = -2Ha^2\frac{1-\cos(n\pi)}{n^3\pi^3}.$$

11. Calculate directly

$$\nabla^2 p = (12Ax^2 + 6Bxy + 2Cy^2) + (2Cx^2 + 6Dxy + 12Ey^2)$$

and equate to $-H = -K(x^2+y^2)$. Then $12A + 2C = -K$, $12E + 2C = -K$. These are two equations in three unknowns A, C, E. Many solutions are possible, especially

$$C = -K/2, A = E = 0: \quad p = -Kx^2y^2/2$$

and

$$C = 0, A = E = -K/12: \quad p = -K(x^4 + y^4)/12$$

Chapter 6

Miscellaneous Exercises

1. Transform the problem

$$U'' - \gamma^2\left(U - \frac{T}{s}\right) = sU - T_0$$

$$U'(0) = 0, \quad U'(1) = 0$$

$$U = \frac{T_0 + \gamma^2 T/s}{\gamma^2 + s} + c_1 \cosh\left(\sqrt{\gamma^2 + s}x\right) + c_2 \sinh\left(\sqrt{\gamma^2 + s}x\right)$$

$U'(0) = 0$ makes $c_2 = 0$; $U'(1) = 0$ makes $c_1 = 0$. Thus U is just the first term. Apply partial fractions

$$\frac{T_0 + \gamma^2 T/s}{\gamma^2 + s} = \frac{T_0}{\gamma^2 + s} + \frac{\gamma^2 T}{s(\gamma^2 + s)} = \frac{T_0}{\gamma^2 + s} + T\left(\frac{1}{s} - \frac{1}{\gamma^2 + s}\right)$$

$$u(t) = (T_0 - T)e^{-\gamma^2 t} + T.$$

The partial differential equation really does not involve x, since there is nothing in the partial differential equation, the initial condition or the boundary conditions that forces u to be different for different values of x.

3. Transform the problem

$$U'' = sU, \quad U'(0) = 0, \quad U(1) = \frac{1}{s^2}.$$

Solution

$$U = \frac{\cosh(\sqrt{s}x)}{s^2 \cosh(\sqrt{s})}.$$

Zeros of the denominator: $s = 0$ and $s = -(n - \frac{1}{2})^2\pi^2 = r_n$, $\quad n = 1, 2, \cdots$. A_0: replace the cosh by the first terms of their Taylor series near 0:

$$U(x,s) \cong \frac{1 + \frac{sx^2}{2} + \cdots}{s^2\left(1 + \frac{s}{2} + \cdots\right)} \cong \frac{\left(1 + \frac{sx^2}{2} + \cdots\right)\left(1 - \frac{s}{2} + \cdots\right)}{s^2} = \frac{1 + \frac{s}{2}(x^2 - 1) + \cdots}{s^2}$$

$$= \frac{1}{s^2} + \frac{x^2 - 1}{2s}.$$

The inverse transform of these terms is $t + \frac{x^2 - 1}{2}$ (known as a heat polynomial) which satisfies the heat equation and the boundary conditions.

A_n: Let

$$q(s) = \frac{\cosh(\sqrt{s}x)}{s^2}, \quad p(s) = \cosh(\sqrt{s}), \quad p' = \frac{1}{2\sqrt{s}}\sinh(\sqrt{s})$$

$$A_n = \frac{\cos((n - \frac{1}{2})\pi x)}{(n - \frac{1}{2})^4\pi^4} \frac{2i(n - \frac{1}{2})\pi}{i\sin((n - \frac{1}{2})\pi)} = \frac{2\cos((n - \frac{1}{2})\pi x)}{(n - \frac{1}{2})^2\pi^3 \sin((n - \frac{1}{2})\pi)}.$$

The solution is

$$u(x,t) = t + \frac{x^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{2\cos(\rho_n x)}{\rho_n^3 \sin(\rho_n)} e^{-\rho_n^2 t}$$

where $\rho_n = (n - \frac{1}{2})\pi$.