INTRODUCTION
TO
LINEAR
ALGEBRA
Fourth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

3 This problem gives the diagonals $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v}=(3,3)$ and $\boldsymbol{w}=(2,-2)$.

4 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=$ (add first answers) $=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination gives $(0,0,0)$. Stated another way: $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero. $c=3$ and $d=9$ give $(3,3,-6)$.

7 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=(0,1,2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Three possible parallelograms!

10 $\boldsymbol{i}-\boldsymbol{j}=(1,1,0)$ is in the base (x-y plane). $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=(1,1,1)$ is the opposite corner from $(0,0,0)$. Points in the cube have $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

11 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Centers of faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 A four-dimensional cube has $2^4=16$ corners and $2\cdot 4=8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum = zero vector. Sum $=-2{:}00$ vector $=8{:}00$ vector. 2:00 is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j}=(0,1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j}=(0,12)$.

15 The point $\dfrac{3}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u}=\dfrac{1}{2}\boldsymbol{v}+\dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V}=-\boldsymbol{v}+2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\ge 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\le c\le 1$ and $0\le d\le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square.

19 With $c\ge 0$ and $d\ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, then the cone is the whole quadrant $x\ge 0$, $y\ge 0$. *Question*: What if $\boldsymbol{w}=-\boldsymbol{v}$? The cone opens to a half-space.

23 You can see why $\boldsymbol{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \boldsymbol{q}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \boldsymbol{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&4\\0&3&6\\0&0&5\end{bmatrix} = QR$.

24 (a) One basis for the subspace $\boldsymbol{S}$ of solutions to $x_1 + x_2 + x_3 - x_4 = 0$ is $\boldsymbol{v}_1 = (1,-1,0,0)$, $\boldsymbol{v}_2 = (1,0,-1,0)$, $\boldsymbol{v}_3 = (1,0,0,1)$ (b) Since $\boldsymbol{S}$ contains solutions to $(1,1,1,-1)^{\mathrm{T}}\boldsymbol{x} = 0$, a basis for $\boldsymbol{S}^{\perp}$ is $(1,1,1,-1)$ (c) Split $(1,1,1,1) = \boldsymbol{b}_1 + \boldsymbol{b}_2$ by projection on $\boldsymbol{S}^{\perp}$ and $\boldsymbol{S}$: $\boldsymbol{b}_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $\boldsymbol{b}_1 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$.

25 This question shows 2 by 2 formulas for QR; breakdown $R_{22} = 0$ when A is singular. $\begin{bmatrix}2&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2&-1\\1&2\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}5&3\\0&1\end{bmatrix}$. Singular $\begin{bmatrix}1&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}2&2\\0&\mathbf{0}\end{bmatrix}$. The Gram-Schmidt process breaks down when $ad - bc = 0$.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

27 When a and b are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. We must use the orthogonal A and B (or orthonormal $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$) to be allowed to add 1D projections.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

29 $\boldsymbol{q}_1 = \frac{1}{3}(2,2,-1)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,-1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(1,-2,-2)$.

30 The columns of the wavelet matrix W are *orthonormal*. Then $W^{-1} = W^{\mathrm{T}}$. See Section 7.2 for more about wavelets : a useful orthonormal basis with many zeros.

31 (a) $c = \frac{1}{2}$ normalizes all the orthogonal columns to have unit length (b) The projection $(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a}$ of $\boldsymbol{b} = (1,1,1,1)$ onto the first column is $\boldsymbol{p}_1 = \frac{1}{2}(-1,1,1,1)$. (Check $\boldsymbol{e} = \mathbf{0}$.) To project onto the plane, add $\boldsymbol{p}_2 = \frac{1}{2}(1,-1,1,1)$ to get $(0,0,1,1)$.

32 $Q_1 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

34 (a) $Q\boldsymbol{u} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$. This is $-\boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ equals 1 (b) $Q\boldsymbol{v} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$.

35 Starting from $\boldsymbol{A} = (1,-1,0,0)$, the orthogonal (not orthonormal) vectors $\boldsymbol{B} = (1,1,-2,0)$ and $\boldsymbol{C} = (1,1,1,-3)$ and $\boldsymbol{D} = (1,1,1,1)$ are in the directions of $\boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$. The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal Q!) are $\begin{bmatrix}\\ \boldsymbol{A}&\boldsymbol{B}&\boldsymbol{C}&\boldsymbol{D}\\ \\ \end{bmatrix} = \begin{bmatrix}1&1&1&1\\-1&1&1&1\\0&-2&1&1\\0&0&-3&1\end{bmatrix}$ and

$$\begin{bmatrix}1&1&1&1&1\\-1&1&1&1&1\\0&-2&1&1&1\\0&0&-3&1&1\\0&0&0&-4&1\end{bmatrix}$$

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) = $ total to find U^{-1}.

8 $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}$;
$A \to \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$ with
$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}$.

9 $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has cofactors $C_{13} = C_{31} = C_{24} = C_{42} = 1$ and $C_{14} = C_{41} = -1$. A^{-1} is a full matrix!

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$.
(b) A has eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of Q: either $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$ **or** $Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$.

12 When A is multiplied by a plane rotation Q_{ij}, this changes the $2n$ (not n^2) entries in rows i and j. Then multiplying on the right by $(Q_{ij})^{-1} = (Q_{ij})^{\mathrm{T}}$ changes the $2n$ entries in columns i and j.

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

14 The $(2,1)$ entry of $Q_{21}A$ is $\frac{1}{3}(-\sin\theta + 2\cos\theta)$. This is zero if $\sin\theta = 2\cos\theta$ or $\tan\theta = 2$. Then the $2, 1, \sqrt{5}$ right triangle has $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$.

Every 3 by 3 rotation with $\det Q = +1$ is the product of 3 plane rotations.

15 This problem shows how elimination is more expensive (the nonzero multipliers are counted by **nnz**(L) and **nnz**(LL)) when we spoil the tridiagonal K by a random permutation.

If on the other hand we start with a poorly ordered matrix K, an improved ordering is found by the code **symamd** discussed in this section.

16 The “red-black ordering” puts rows and columns 1 to 10 in the odd-even order $1, 3, 5, 7, 9, 2, 4, 6, 8, 10$. When K is the $-1, 2, -1$ tridiagonal matrix, odd points are connected