

Student Solutions Manual
for use with
**Complex Variables
and Applications**
Seventh Edition

Selected Solutions to Exercises in Chapters 1-7

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(b) In the same way, we can show that

$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n \quad (n = 2, 3, \dots).$$

This is true when $n = 2$ (Sec. 5). Assuming that it is true when $n = m$, we write

$$\begin{aligned} \overline{z_1 z_2 \cdots z_m z_{m+1}} &= \overline{(z_1 z_2 \cdots z_m) z_{m+1}} = \overline{(z_1 z_2 \cdots z_m)} \bar{z}_{m+1} \\ &= (\bar{z}_1 \bar{z}_2 \cdots \bar{z}_m) \bar{z}_{m+1} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_m \bar{z}_{m+1}. \end{aligned}$$

14. The identities (Sec. 5) $z\bar{z} = |z|^2$ and $\operatorname{Re} z = \frac{z + \bar{z}}{2}$ enable us to write $|z - z_0| = R$ as

$$(z - z_0)(\bar{z} - \bar{z}_0) = R^2,$$

$$z\bar{z} - (z\bar{z}_0 + \bar{z}\bar{z}_0) + z_0\bar{z}_0 = R^2,$$

$$|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

15. Since $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$, the hyperbola $x^2 - y^2 = 1$ can be written in the following ways:

$$\left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 = 1,$$

$$\frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4} = 1,$$

$$\frac{2z^2 + 2\bar{z}^2}{4} = 1,$$

$$z^2 + \bar{z}^2 = 2.$$

SECTION 7

1. (a) Since

$$\arg\left(\frac{i}{-2-2i}\right) = \arg i - \arg(-2-2i),$$

one value of $\arg\left(\frac{i}{-2-2i}\right)$ is $\frac{\pi}{2} - \left(-\frac{3\pi}{4}\right)$, or $\frac{5\pi}{4}$. Consequently, the principal value is

$$\frac{5\pi}{4} - 2\pi, \text{ or } -\frac{3\pi}{4}.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

So there is no neighborhood throughout which f is analytic, and this means that $\cos \bar{z}$ is nowhere analytic.

16. (a) Use expression (12), Sec. 33, to write

$$\overline{\cos(iz)} = \overline{\cos(-y + ix)} = \cos y \cosh x - i \sin y \sinh x$$

and

$$\cos(i\bar{z}) = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x.$$

This shows that $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z .

(b) Use expression (11), Sec. 33, to write

$$\overline{\sin(iz)} = \overline{\sin(-y + ix)} = -\sin y \cosh x - i \cos y \sinh x$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Evidently, then, the equation $\overline{\sin(iz)} = \sin(i\bar{z})$ is equivalent to the pair of equations

$$\sin y \cosh x = 0, \quad \cos y \sinh x = 0.$$

Since $\cosh x$ is never zero, the first of these equations tells us that $\sin y = 0$. Consequently, $y = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$). Since $\cos n\pi = (-1)^n \neq 0$, the second equation tells us that $\sinh x = 0$, or that $x = 0$. So we may conclude that $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = 0 + in\pi = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$).

17. Rewriting the equation $\sin z = \cosh 4$ as $\sin x \cosh y + i \cos x \sinh y = \cosh 4$, we see that we need to solve the pair of equations

$$\sin x \cosh y = \cosh 4, \quad \cos x \sinh y = 0$$

(c) Note that

$$\frac{(s^2 - 1)^n}{(s - 1)^{n+1}} = \frac{(s - 1)^n (s + 1)^n}{(s - 1)^{n+1}} = \frac{(s + 1)^n}{s - 1}.$$

Referring to the final result in part (b), then, we have

$$P_n(1) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s - 1)^{n+1}} ds = \frac{1}{2^n} \frac{1}{2 \pi i} \int_C \frac{(s + 1)^n}{s - 1} ds = \frac{1}{2^n} 2^n = 1 \quad (n = 0, 1, 2, \dots).$$

Also, since

$$\frac{(s^2 - 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n (s + 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n}{s + 1},$$

we have

$$P_n(-1) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s + 1)^{n+1}} ds = \frac{1}{2^n} \frac{1}{2 \pi i} \int_C \frac{(s - 1)^n}{s + 1} ds = \frac{1}{2^n} (-2)^n = (-1)^n \quad (n = 0, 1, 2, \dots).$$

9. We are asked to show that

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s) ds}{(s - z)^3}.$$

(a) In view of the expression for $f'(z)$ in the lemma,

$$\begin{aligned} \frac{f'(z + \Delta z) - f'(z)}{\Delta z} &= \frac{1}{2 \pi i} \int_C \left[\frac{1}{(s - z - \Delta z)^2} - \frac{1}{(s - z)^2} \right] \frac{f(s) ds}{\Delta z} \\ &= \frac{1}{2 \pi i} \int_C \frac{2(s - z) - \Delta z}{(s - z - \Delta z)^2 (s - z)^2} f(s) ds. \end{aligned}$$

Then

$$\begin{aligned} \frac{f'(z + \Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i} \int_C \frac{f(s) ds}{(s - z)^3} &= \frac{1}{2 \pi i} \int_C \left[\frac{2(s - z) - \Delta z}{(s - z - \Delta z)^2 (s - z)^2} - \frac{2}{(s - z)^3} \right] f(s) ds \\ &= \frac{1}{2 \pi i} \int_C \frac{3(s - z) \Delta z - 2(\Delta z)^2}{(s - z - \Delta z)^2 (s - z)^3} f(s) ds. \end{aligned}$$

(b) When $f(z) = \frac{1}{1+z^2}$, we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - \dots \quad (0 < |z| < 1).$$

Thus

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(0) = 0.$$

(c) If $f(z) = \frac{1}{z}$, it follows that $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z}$. Evidently, then,

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(1) = 2\pi i.$$

4. Let C denote the circle $|z|=1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ ($|z| < \infty$) enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z} dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!} dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \quad (n = 0, 1, 2, \dots).$$

Now the $\frac{1}{z}$ in this series occurs when $n-k=-1$, or $k=n+1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \quad (n = 0, 1, 2, \dots).$$

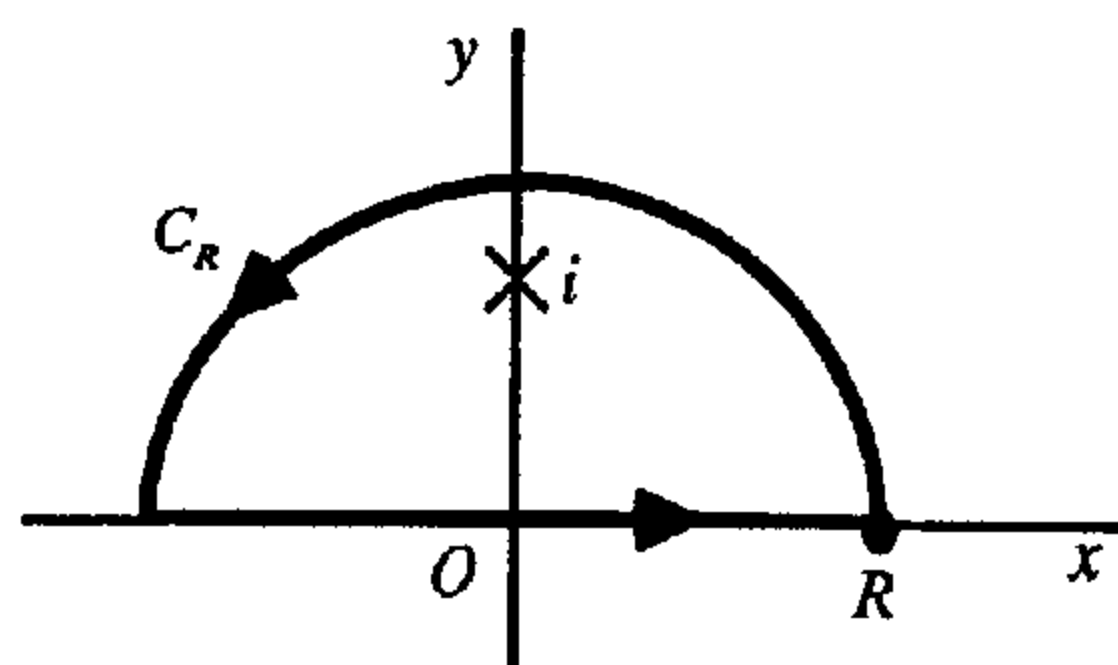
The final result in part (a) thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

Chapter 7

SECTION 72

1. To evaluate the integral $\int_0^{\infty} \frac{dx}{x^2+1}$, we integrate the function $f(z) = \frac{1}{z^2+1}$ around the simple closed contour shown below, where $R > 1$.



We see that

$$\int_{-R}^R \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1} = 2\pi i B,$$

where

$$B = \operatorname{Res}_{z=i} \frac{1}{z^2+1} = \operatorname{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \frac{1}{z+i} \Big|_{z=i} = \frac{1}{2i}.$$

Thus

$$\int_{-R}^R \frac{dx}{x^2+1} = \pi - \int_{C_R} \frac{dz}{z^2+1}.$$

Now if z is a point on C_R ,

$$|z^2+1| \geq ||z|^2-1| = R^2-1;$$

and so

$$\left| \int_{C_R} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{R^2-1} = \frac{\frac{\pi}{R}}{1-\frac{1}{R^2}} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Finally, then

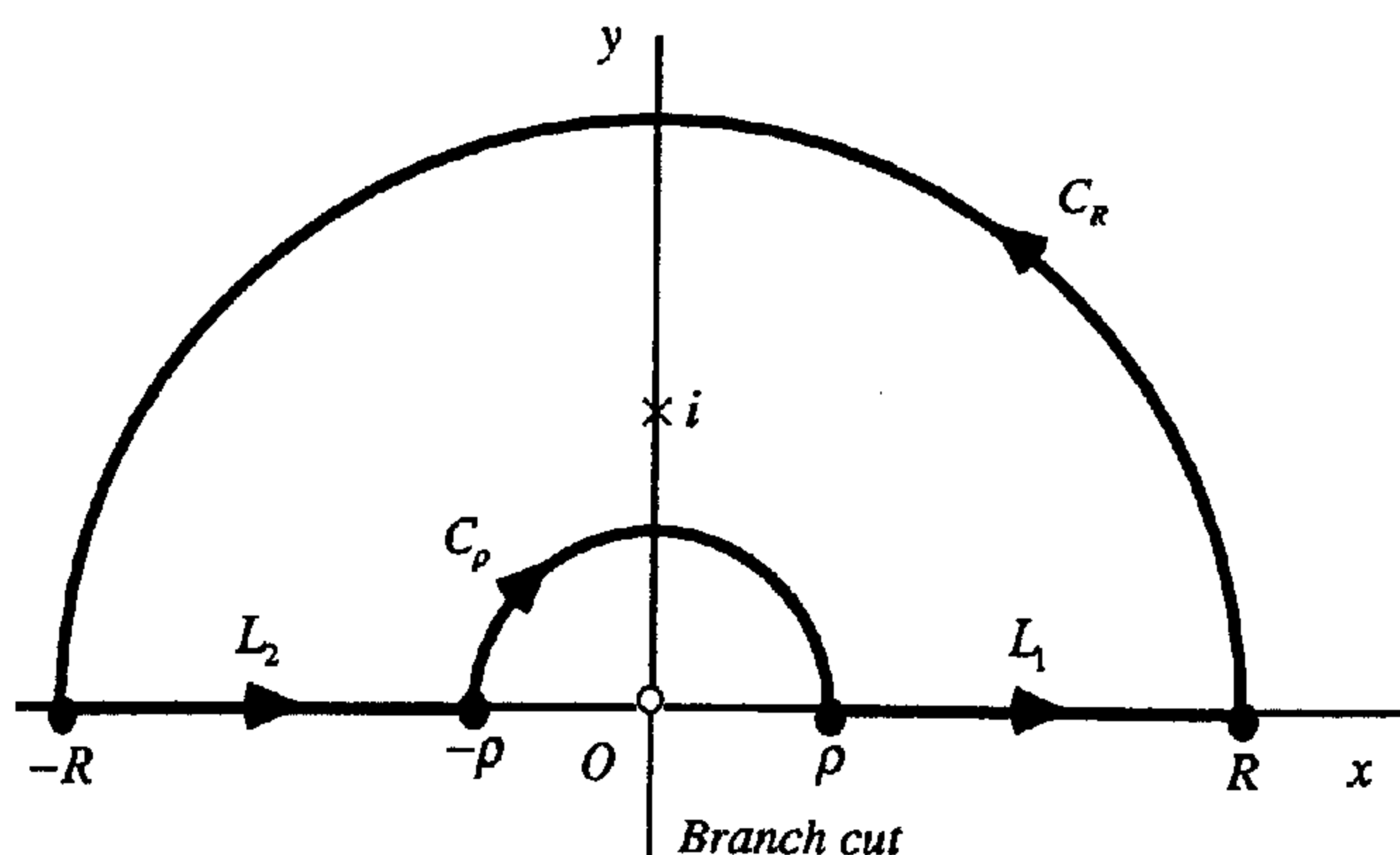
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi, \quad \text{or} \quad \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

6. (a) Let us first use the branch

$$f(z) = \frac{z^{-1/2}}{z^2 + 1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2 + 1} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2 + 1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z) dz + \int_{C_R} f(z) dz + \int_{L_2} f(z) dz + \int_{C_\rho} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Since

$$L_1: z = re^{i0} = r \quad (\rho \leq r \leq R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \quad (\rho \leq r \leq R),$$

we may write

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} - i \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} = (1 - i) \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)}.$$

Thus

$$(1 - i) \int_{\rho}^R \frac{dr}{\sqrt{r}(r^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$